"Sing and Dance!"
Input/Output Logics without Weakening

Xavier Parent and Leendert van der Torre

University of Luxembourg
x.parent@mail.com
leonvandertorre@uni.lu

Abstract. Makinson and van der Torre [14] introduce a number of input/output (I/O) logics to reason about conditional norms. The key idea is to make obligations relative to a given set of conditional norms. The meaning of the normative concepts is, then, given in terms of a set of procedures yielding outputs for inputs. Using the same methodology, Stolpe [20,21] has developed some more I/O logics to include systems without the rule of weakening of the output (or principle of inheritance). We extend Stolpe's account in two directions. First, we show how to make it support reasoning by cases−a common form of reasoning. Second, we show how to inject a new (as we call it, "aggregative") form of cumulative transitivity, which we think is more suitable for normative reasoning. The main outcomes of the paper are soundness and completeness theorems for the proposed systems with respect to their intended semantics.

1 Introduction

Makinson and van Torre [14] introduce a number of input/output (I/O) logics to reason about conditional norms. The key idea is to make obligations relative to a given set of conditional norms. The meaning of the normative concepts is, then, given in terms of a set of procedures yielding outputs for inputs. A number of I/O operations are studied in the aforementioned paper [14]. It is shown that they correspond to a series of proof systems of increasing strength. I/O logic promotes a paradigm shift from modal logic to what has recently been called "norm-based semantics" by Hansen [10, p. 288]. The core idea is to explain the truths of deontic logic, not by some set of possible worlds among which some are ideal or at least better than others, but with reference to an explicit set of given norms or existing moral standards. The founders of I/O logic mostly criticized the modal logic paradigm for−to use Quine's famous expression−having been "conceived in the sin": the sin of assuming that norms bear truth-values. Thus, their main motivation was philosophical. Still, one reason why modal logic has been so popular in deontic logic is that it is a general framework, which provides us with plenty of freedom to pick and choose the axiom schemata we think are right. Whatever philosophical reservations one may have about the use of modal logic in deontic logic, one would like to know if, or to what extent, norm-based semantics in general, and I/O logic in particular, can offer the same kind of flexibility.

In this paper, we focus on the so-called rule of Weakening of the Output (WO), which all the I/O operations defined by Makinson and van Torre [14] satisfy. The rule may be given the following form, where the conditional obligation for x given a is written as (a, x), and $\vdash$ stands for the deducibility relation in propositional logic:

$$\text{WO}\ \frac{(a,x) \qquad x \vdash y}{(a,y)}$$

This is also known as the "principle of inheritance". It has been called into question, mostly in connection with the deontic paradoxes [5,6,11,9,2] and the question of how to accommodate conflicts between obligations−see, e.g., [7,8]. This raises the question whether the framework may be generalized to include systems without output weakening; if yes, how.

A first step towards answering the above question was made by Stolpe [20,21]. He considers two of the four standard I/O operations defined by Makinson and van der Torre [14], namely the so-called simple-minded I/O operation out_1, and the so-called reusable I/O operation out_3. Both develop output by detachment. While out_1 spells out the basic mechanism used to achieve this, out_3 extends it to cover iteration of successive detachments. For both operations, a suitable semantics is given, for which the rule WO fails. Each semantics comes with a sound and complete axiomatic characterization. The present paper extends Stolpe's account in two ways.

First, we show how to refine Stolpe's account to make the latter one support reasoning by cases−this is a common form of reasoning. We look at the I/O operation out_2, called "basic" by Makinson and van der Torre [14]. Its distinctive feature is that it validates the rule OR:

$$\text{OR}\ \frac{(a,x) \qquad (b,x)}{(a \vee b, x)}$$

The present paper shows how to incorporate such a rule. We provide a suitable semantics for the I/O operation, a proof system for it, and a completeness result linking the two.

Second, we show how to integrate other forms of cumulative transitivity. There is no doubt that some form of transitivity is required for an adequate account of norms and normative systems. One reason is that transitivity serves as a means of binding together different parts of a code. For instance, a legal system is invariably organized into different modules which are interrelated. Rules from one module stipulate legal consequences that are used as premises for some rules part of another module. Penal law may, e.g., state that grand larseny ought to be added to a person's criminal record, whereas administrative law may stipulate that an unblemished record is a prerequisite for public office. Some form of transitivity is required to go from grand larseny to the bar to holding public office.

Stolpe uses the rule of (as he calls it) “mediated cumulative transitivity” (MCT):

$$\text{MCT}\ \frac{(a, x') \qquad x' \vdash x \qquad (a \wedge x, y)}{(a, y)}$$

As we will see in Section 4, given the other rules of his system, MCT turns out to be equivalent to the rule of cumulative transitivity (CT), as initially used by Makinson and van der Torre [14]:

$$\text{CT}\ \frac{(a, x) \qquad (a \wedge x, y)}{(a, y)}$$

We look at the following alternative (call it “aggregative”) variant, first introduced in a companion paper [17]:

$$\text{ACT}\ \frac{(a, x) \qquad (a \wedge x, y)}{(a, x \wedge y)}$$

The counterexamples usually given to CT in the literature [16,12,13] no longer work, when ACT is used in place of CT. This is because they all rely on the intuition that the obligation of y ceases to hold when the obligation of (a, x) is violated. The following example, due to Broome [1, § 7.4], may be used to illustrate this point:

You ought to exercise hard everyday	$(\top, x)$
If you exercise hard everyday, you ought to eat heartily	(x, y)
?* You ought to eat heartily	?* $(\top, y)$

Intuitively, the obligation to eat heartily no longer holds, if you take no exercise. In this example, the correct conclusion is $(\top, x \wedge y)$, and not $(\top, y)$. Thus, ACT appears to be more suitable for normative reasoning, because it keeps track of what has been previously detached.

The layout of this paper is as follows. Section 2 lays the groundwork, tackling out_1 in essentially the same way as Stolpe does. Section 3 extends the account so that reasoning by cases is supported. Section 4 shows how to inject the aggregative form of cumulative transitivity mentioned above. The main achievement of the paper is the establishment of soundness and completeness theorems for the proposed systems with respect to their intended semantics. We do not give all the details of the soundness and completeness proofs, but we outline the main steps.[1] Section 5 discusses some properties satisfied by the I/O operations defined in this paper.

[1] The detailed proofs will be included in the journal version of the present paper.

2 Developing the Output by Detachment (out_1)

We start with the simple-minded I/O operation out_1. The I/O operation to be defined here is noted $\mathcal{O}_1$. It is essentially a variation on the I/O operation PN_1 put forth by Stolpe [20,21]. The main reason for including such an operation in our study is that the completeness result for it will be needed for subsequent developments.

First, some definitions are needed. A normative code is a set N of conditional obligations. A conditional obligation is a pair $(a,\ x)$, where a and x are two formulae of classical propositional logic. We use this notation instead of $\bigcirc(x \mid a)$, because the latter has distinct interpretations in the literature. In the notation $(a,\ x)$, the first element a is called the body of the rule, and is thought of as an input, representing some condition or situation. The second element x is called the head of the rule, and is thought of as an output, representing what the norm tells us to be obligatory in that situation. We use the standard notation $(\top, x)$ for the unconditional obligation of x, where $\top$ is a zero-place connective standing for 'tautology'. In I/O logic, the main construct has the form

$$x \in out(N, a)$$

Intuitively: given input a (state of affairs), x (obligation) is in the output under norms N. An equivalent notation is: $(a, x) \in out(N)$. The I/O operations to be defined in this paper will be denoted by the symbol $\mathcal{O}$ in order to avoid any confusion with *out* and $\bigcirc$.

Some further notation. $\mathcal{L}$ is the set of all formulae of classical propositional logic. Given an input $A \subseteq \mathcal{L}$, and a set N of norms, $N(A)$ denotes the image of N under A, i.e., $N(A) = \{x : (a, x) \in N \text{ for some } a \in A\}$. $Cn(A)$ denotes the set $\{x : A \vdash x\}$, where $\vdash$ is the deducibility relation used in classical propositional logic. The notation $x \dashv\vdash y$ is short for $x \vdash y$ and $y \vdash x$. We use PL as an abbreviation for (classical) propositional logic. Given $M \subseteq N$, we denote by $h(M)$ the set of all the heads of elements of M, viz $h(M) = \{x : (a, x) \in M\}$.

Definition 1 (Semantics). *$x \in \mathcal{O}_1(N, A)$ if and only if there is some finite $M \subseteq N$ such that*

- *$M(Cn(A)) \neq \emptyset$, and*
- *$x \dashv\vdash \wedge M(Cn(A))$*

Intuitively: x is equivalent to the conjunction of heads of rules in some $M \subseteq N$ that are all triggered by input A.

The main difference between $\mathcal{O}_1$ and PN_1 arises when A does not trigger any norm, viz. $M(Cn(A)) = \emptyset$ for all $M \subseteq N$. In this limiting case, PN_1 outputs the set of all tautologies, while $\mathcal{O}_1$ outputs nothing. Von Wright [22, pp. 152-4] argues, rightly in our view, that the obligation of $\top$ does not express a genuine prescription.

$\mathcal{O}_1$ is monotonic with respect to the input set. The latter claim requires a careful and detailed proof, because there is a pitfall to avoid.

Theorem 1 (Factual monotony). *We have* $\mathcal{O}_1(N, A) \subseteq \mathcal{O}_1(N, B)$ *whenever* $Cn(A) \subseteq Cn(B)$.

Proof. Assume $x \in \mathcal{O}_1(N, A)$ and $Cn(A) \subseteq Cn(B)$. From the former, there is some finite $M_1 \subseteq N$ such that $M_1(Cn(A)) \neq \emptyset$, and

1. $x \dashv\vdash \wedge M_1(Cn(A))$

There is no guarantee that input set B does not trigger more pairs in M_1 than A does. To circumvent this problem, the argument takes a detour through the set

$$M_1^- = \{(c, y) \in M_1 : c \in Cn(A)\}$$

Thus, M_1^- is M_1 "stripped of" all the pairs that are not triggered by A. We have $M_1(Cn(A)) = M_1^-(Cn(A))$. We also have $M_1^-(Cn(A)) = M_1^-(Cn(B))$, viz.

$$\{y : (c, y) \in M_1^-, c \in Cn(A)\} = \{y : (c, y) \in M_1^-, c \in Cn(B)\}$$

The $\subseteq$-direction follows from the second opening assumption, $Cn(A) \subseteq Cn(B)$. The $\supseteq$-direction follows from the definition of M_1^-. The argument may, then, be continued thus:

2. $x \dashv\vdash \wedge M_1^-(Cn(A))$
3. $x \dashv\vdash \wedge M_1^-(Cn(B))$

Thus, $x \in \mathcal{O}_1(N, B)$ as required. □

It immediately follows that $\mathcal{O}_1(N, A) \subseteq \mathcal{O}_1(N, B)$ whenever $A \subseteq B$.

We set $\mathcal{O}_1(N) = \{(A, x) : x \in \mathcal{O}_1(N, A)\}$. The notion of derivation is defined as in standard I/O logic except that $(\top, \top)$ is not allowed to appear in a derivation unless it is explicitly given in the set N of assumptions.

Definition 2 (Proof system). $(a, x) \in \mathcal{D}_1(N)$ *if and only if there is a derivation of* (a, x) *from* N *using the rules {SI, EQ, AND}:*

$$SI\ \frac{(a, x) \qquad b \vdash a}{(b, x)} \qquad\qquad EQ\ \frac{(a, x) \qquad x \dashv\vdash y}{(a, y)}$$

$$AND\ \frac{(a, x) \qquad (a, y)}{(a, x \wedge y)}$$

Where A is a set of formulae, $(A, x) \in \mathcal{D}_1(N)$ means that $(a, x) \in \mathcal{D}_1(N)$, for some conjunction a of formulae, all taken from a finite subset of A. $\mathcal{D}_1(N, A)$ is $\{x : (A, x) \in \mathcal{D}_1(N)\}$.

Theorem 2. $\mathcal{O}_1$ *validates the rules of* $\mathcal{D}_1$ *(for individual formulae a).*

Proof. The argument is straightforward, and left to the reader. (For SI, the same trick as in the proof of Theorem 1 must be used.) □

Theorem 3 (Soundness). $\mathcal{D}_1(N, A) \subseteq \mathcal{O}_1(N, A)$

Proof. The proof is by induction on the length of the derivation, using Theorems 1 and 2. □

Theorem 4 (Completeness). $\mathcal{O}_1(N, A) \subseteq \mathcal{D}_1(N, A)$

Proof. Assume $x \in \mathcal{O}_1(N, A)$. So there exists some finite $M \subseteq N$ such that $M(Cn(A)) = \{x_1, ..., x_n\} \neq \emptyset$ and $x \dashv\vdash \wedge_{i=1}^n x_i$. For each x_i, there is some $a_i \in Cn(A)$ such that $(a_i, x_i) \in M$. For each a_i, there is also a conjunction b_i of elements in A such that $b_i \vdash a_i$. A derivation of (A, x) from M, and hence from N, is shown below.

$$\dfrac{\dfrac{\dfrac{(a_1, x_1)}{(\wedge_{i=1}^n b_i, x_1)}\text{SI} \quad \cdots\cdots \quad \dfrac{(a_n, x_n)}{(\wedge_{i=1}^n b_i, x_n)}\text{SI}}{(\wedge_{i=1}^n b_i, \wedge_{i=1}^n x_i)}\text{AND}}{(\wedge_{i=1}^n b_i, x)}\text{EQ}$$

This is a derivation of (A, x), as $\wedge_{i=1}^n b_i$ is a conjunction of elements in A. □

3 Reasoning by Cases

In this section, the account described in the previous section is extended to the basic operation out_2, which supports reasoning by cases. The I/O operation is denoted $\mathcal{O}_2$, and the corresponding proof system is called $\mathcal{D}_2$. We call a set of formulae complete if it is either equal to $\mathcal{L}$ or maximal consistent (the set is consistent, and none of its proper extensions is consistent).

Definition 3. $\mathcal{O}_2(N, A) = \cap\{\mathcal{O}_1(N, V) : A \subseteq V, V\, complete\}$.

Theorem 5. $\mathcal{O}_1(N, A) \subseteq \mathcal{O}_2(N, A)$.

Proof. Let $x \in \mathcal{O}_1(N, A)$. Let V be a complete set such that $A \subseteq V$. By Theorem 1, $x \in \mathcal{O}_1(N, V)$. By Definition 3, $x \in \mathcal{O}_2(N, A)$ as required. □

Theorem 6 (Factual monotony). $\mathcal{O}_2(N, A) \subseteq \mathcal{O}_2(N, B)$ *if* $Cn(A) \subseteq Cn(B)$

Proof. Assume $x \in \mathcal{O}_2(N, A)$ and $Cn(A) \subseteq Cn(B)$. Let V be a complete set such that $B \subseteq V$. We have $Cn(B) \subseteq Cn(V) = V$. From this and the second opening assumption, $Cn(A) \subseteq V$. So, $A \subseteq V$. From this and the first opening assumption, $x \in \mathcal{O}_1(N, V)$. Thus, $x \in \mathcal{O}_2(N, B)$. □

Definition 4. $(a, x) \in \mathcal{D}_2(N)$ *if and only if there is a derivation of* (a, x) *from* N *using the rules of* $\mathcal{D}_1$ *supplemented with*

$$OR\ \dfrac{(a, x) \qquad (b, x)}{(a \vee b, x)}$$

The next theorem appeals to the fact that $\mathcal{O}_1$ validates AND and EQ for an input set of arbitrary cardinality rather than just a singleton set. The argument is virtually the same in both cases. Details are omitted.

Theorem 7. *$\mathcal{O}_2$ validates the rules of $\mathcal{D}_2$ (for individual formulae a).*

Proof. For SI. Assume $x \in \mathcal{O}_2(N, a)$ with $b \vdash a$. Let V be a complete set such that $b \in V$. From $b \vdash a$, we get $a \in V$. By Definition 3, we infer $x \in \mathcal{O}_1(N, V)$. This shows that $x \in \mathcal{O}_2(N, b)$.

For AND. Assume $x \in \mathcal{O}_2(N, a)$ and $y \in \mathcal{O}_2(N, a)$. Let V be a complete set such that $a \in V$. By Definition 3, $x \in \mathcal{O}_1(N, V)$ and $y \in \mathcal{O}_1(N, V)$. Since $\mathcal{O}_1$ validates AND, $x \wedge y \in \mathcal{O}_1(N, V)$. This shows that $x \wedge y \in \mathcal{O}_2(N, a)$.

For OR. Assume $x \in \mathcal{O}_2(N, a)$ and $x \in \mathcal{O}_2(N, b)$. Let V be a complete set containing $a \vee b$. Since V is complete, either $a \in V$ or $b \in V$. Assume that the first applies. In that case, $x \in \mathcal{O}_1(N, V)$, by the first opening assumption and Definition 3. Assume the second applies. In that case $x \in \mathcal{O}_1(N, V)$, by the second opening assumption and Definition 3. Either way, $x \in \mathcal{O}_1(N, V)$, and thus $x \in \mathcal{O}_2(N, a \vee b)$ as required.

For EQ, assume $x \in \mathcal{O}_2(N, a)$ and $x \dashv\vdash y$. Let V be a complete set containing a. By Definition 3, $x \in \mathcal{O}_1(N, V)$. Since $\mathcal{O}_1$ validates EQ, $y \in \mathcal{O}_1(N, V)$, and so $y \in \mathcal{O}_2(N, a)$ as required. □

Theorem 8 (Soundness). *$\mathcal{D}_2(N, A) \subseteq \mathcal{O}_2(N, A)$.*

Proof. Same argument as before, but using Theorems 6 and 7. □

Theorem 9 (Completeness). *$\mathcal{O}_2(N, A) \subseteq \mathcal{D}_2(N, A)$.*

Proof. We give an outline of the proof for a singleton input set $\{a\}$. The proof may easily be generalized to an input set of arbitrary cardinality. For ease of exposition, throughout the proof we write (SI,AND) to indicate an application of SI followed by that of AND. We break the argument into two cases.

Case 1: a is inconsistent. In this case, there is exactly one complete set V containing a; it is $\mathcal{L}$. So $\mathcal{O}_2(N, a) = \mathcal{O}_1(N, \mathcal{L})$. Let $x \in \mathcal{O}_1(N, \mathcal{L})$. This means that $x \dashv\vdash \wedge_{i=1}^{n} x_i$, for $x_1, ..., x_n \in h(N)$. Let $a_1, ..., a_n$ be the body of the rules in question. We have $a \vdash \wedge_{i=1}^{n} a_i$. A derivation of (a, x) from N may, then, be obtained as shown below.

$$\text{SI}\ \dfrac{\dfrac{\dfrac{(a_1, x_1) \quad ... \quad (a_n, x_n)}{(\wedge_{i=1}^{n} a_i, \wedge_{i=1}^{n} x_i)}\ \text{(SI,AND)} \qquad \wedge_{i=1}^{n} x_i \dashv\vdash x}{(\wedge_{i=1}^{n} a_i, x)}\ \text{EQ} \qquad a \vdash \wedge_{i=1}^{n} a_i}{(a, x)}$$

Case 2: a is consistent. Assume (for reductio) that $x \in \mathcal{O}_2(N, a)$ and that $x \notin \mathcal{D}_2(N, a)$. From the former, $x \dashv\vdash \wedge_{i=1}^{n} x_i$, for $x_1, ..., x_n \in h(N)$. In order to derive the contradiction that $x \notin \mathcal{O}_2(N, a)$, we start by showing that $\{a\}$ can be extended to some "maximal" $V \supseteq \{a\}$ such that $x \notin \mathcal{D}_2(N, V)$. By maximal, we mean that for all $V' \supset V$, $x \in \mathcal{D}_2(N, V')$. Thus, V is amongst the "biggest" input sets V containing a and not making x derivable.

V is built from a sequence of sets $V_0, V_1, V_2, ...$ as follows. Consider an enumeration $x_1, x_2, x_3, ...$ of all the formulae. We define:

$$V_0 = \{a\}$$
$$V_n = \begin{cases} V_{n-1} \cup \{x_n\}, & \text{if } x \notin \mathcal{D}_2(N, V_{n-1} \cup \{x_n\}) \\ V_{n-1}, & \text{otherwise} \end{cases}$$
$$V = \cup\{V_n : n \geq 0\}$$

It is a straightforward matter to show the following:

Fact 1 *$x \notin \mathcal{D}_2(N, V_n)$, for all $n \geq 0$.*

Fact 2 *$V_n \subseteq V$, for all $n \geq 0$.*

Fact 3 *For every finite subset $V' \subseteq V$, $V' \subseteq V_n$, for some $n \geq 0$.*

By Fact 2, V includes $\{a\}$ (=V_0).The argument may be continued thus:

Claim 1 *$x \notin \mathcal{D}_2(N, V)$.*

Proof of the claim. Assume, to reach a contradiction, that $x \in \mathcal{D}_2(N, V)$. By compactness for $\mathcal{D}_2$, $x \in \mathcal{D}_2(N, V')$ for some finite $V' \subseteq V$. By Fact 3, $V' \subseteq V_n$ for some $n \geq 0$. By monotony in the right argument, $x \in \mathcal{D}_2(N, V_n)$. This contradicts Fact 1.

Claim 2 *For all $V' \supset V$, $x \in \mathcal{D}_2(N, V')$.*

Proof of the claim. Let $V' \supset V$. So, there is some y such that $y \in V'$ but $y \notin V$. Any such y is such that $y = x_n$, for some $n \geq 1$. By Fact 2, $V_n \subseteq V$. So, $y \notin V_n$. By construction, $V_{n-1} = V_n$, and $x \in \mathcal{D}_2(N, V_{n-1} \cup \{y\}) = \mathcal{D}_2(V, V_n \cup \{y\})$. But $V_n \cup \{y\} \subseteq V \cup \{y\} \subseteq V'$. By monotony in the right argument for $\mathcal{D}_2$, we get that $x \in \mathcal{D}_2(N, V')$, as required.

Claim 3 *V is consistent.*

Proof of the claim. Assume not. Since $x \dashv\vdash \wedge_{i=1}^{n} x_i$, for $x_1, ..., x_n \in h(N)$, a derivation of (V, x) from N may be obtained by reiterating the argument under case 1, contradicting Claim 1.

Claim 4 *V is $\neg$-complete; that is, for all y, either $y \in V$ or $\neg y \in V$.*

Proof of the claim. Assume $y \notin V$ and $\neg y \notin V$ for some y. By Claim 2, it follows that $x \in \mathcal{D}_2(N, V \cup \{y\})$ and $x \in \mathcal{D}_2(N, V \cup \{\neg y\})$. Thus, $(b \wedge y, x)$ and $(c \wedge \neg y, x)$ are both derivable from N, where b and c are conjunctions of elements of V. The following is, then, derivable:

$$\cfrac{\cfrac{(b \wedge y, x) \qquad (c \wedge \neg y, x)}{((b \wedge y) \vee (c \wedge \neg y), x)}\text{OR}}{(b \wedge c, x)}\text{SI}$$

Thus, $x \in \mathcal{D}_2(N, V)$, in contradiction with Claim 1.

Claim 5 *V is maximal consistent; that is, if $V \cup \{y\}$ is consistent, then $y \in V$.*

Proof of the claim. Assume $y \notin V$. By Claim 4, $\neg y \in V$. It, then, follows that $V \cup \{y\}$ is inconsistent, as required.

We are almost finished. By Theorem 3 and Theorem 4, we have $\mathcal{O}_1(N, V) = \mathcal{D}_1(N, V) \subseteq \mathcal{D}_2(N, V)$. So $x \notin \mathcal{O}_1(N, V)$. Hence, $x \notin \mathcal{O}_2(N, A)$. □

4 Aggregative Cumulative Transitivity

This section shows how to redefine Makinson and van der Torre's reusable output operation out_3 so that it validates neither WO nor CT but ACT:

$$\text{ACT}\ \frac{(a, x) \qquad (a \wedge x, y)}{(a, x \wedge y)} \qquad\qquad \text{CT}\ \frac{(a, x) \qquad (a \wedge x, y)}{(a, y)}$$

ACT and WO together imply CT.

Stolpe [20,21] named "PN_3" his own variant of out_3. The distinctive rule of PN_3 is the rule MCT mentioned in the introduction:

$$\text{MCT}\ \frac{(a, x') \qquad x' \vdash x \qquad (a \wedge x, y)}{(a, y)}$$

We said that, given the other rules in Stolpe's system, MCT is equivalent to CT. This is easily checked. The other rules are: SI, AND and EQ. On the one hand, given reflexivity for $\vdash$, MCT entails CT. For assume (a, x) and $(a \wedge x, y)$. Since $x \vdash x$, a direct application of MCT yields (a, y). On the other hand, given SI, CT entails MCT:

$$\text{CT}\ \cfrac{(a, x') \qquad \cfrac{(a \wedge x, y) \qquad \cfrac{x' \vdash x}{a \wedge x' \vdash a \wedge x}}{(a \wedge x', y)}\text{SI}}{(a, y)}$$

Note that, given SI, AND and EQ (we will keep them all), ACT is equivalent to:

$$\text{AMCT}\ \frac{(a, x') \qquad x' \vdash x \qquad (a \wedge x, y)}{(a, x' \wedge y)}$$

In this respect, weakening has still a "ghostly" role to play for iteration of successive detachments.

For the sake of conciseness, throughout this section $\mathcal{B}_A^M$ will denote the set of all the Bs such that $A \subseteq B = Cn(B) \supseteq M(B)$. Intuitively, $\mathcal{B}_A^M$ gathers all the Bs that contain A and are closed under both Cn and M.

Definition 5 (Semantics). *$x \in \mathcal{O}_3(N, A)$ if and only if there is some finite $M \subseteq N$ such that,*

- *$M(Cn(A)) \neq \emptyset$, and*
- *for all B, if $B \in \mathcal{B}_A^M$, then $x \dashv\vdash \wedge M(B)$.*

We do not single out any particular B as "proper". But we highlight two very useful such Bs, which we call the smallest and the largest: $\cap \mathcal{B}_A^M$; $\mathcal{L}$.

A subset M of N that makes $x \in \mathcal{O}_3(N, A)$ true is called an "A-witness for x". Unlike with $\mathcal{O}_1$, we have the guarantee that such a M does not contain any rule that is superfluous, viz. not required to get output x:

Theorem 10. *If M is an A-witness for x, then $x \dashv\vdash \wedge h(M)$.*

Proof. Let M be an A-witness for x. By Definition 5, $M(Cn(A)) \neq \emptyset$, and $x \dashv\vdash \wedge M(B)$ for all $B \in \mathcal{B}_A^M$. Consider $B = \mathcal{L}$. We have $x \dashv\vdash \wedge M(\mathcal{L})$. But $M(\mathcal{L}) = h(M)$, and thus $x \dashv\vdash \wedge h(M)$. □

Theorem 11 (Factual monotony). *We have $\mathcal{O}_3(N, A_1) \subseteq \mathcal{O}_3(N, A_2)$ whenever $Cn(A_1) \subseteq Cn(A_2)$.*

Proof. Assume $x \in \mathcal{O}_3(N, A_1)$ and $Cn(A_1) \subseteq Cn(A_2)$. From the first, we get: there is some finite $M_1 \subseteq N$ such that $M_1(Cn(A_1)) \neq \emptyset$ and, for all $B \in \mathcal{B}_{A_1}^{M_1}$,

$$M_1(B) = \{x_1, ..., x_n\} \text{ and } x \dashv\vdash \wedge_{i=1}^n x_i \tag{1}$$

Note that, by Theorem 10, $x \dashv\vdash \wedge h(M_1)$, and so the trick used for the proof of Theorem 1 is no longer needed.

From $Cn(A_1) \subseteq Cn(A_2)$, we get $M_1(Cn(A_1)) \subseteq M_1(Cn(A_2))$. Therefore, $M_1(Cn(A_2)) \neq \emptyset$. Now, consider some $B_1 \in \mathcal{B}_{A_2}^{M_1}$. We have $A_2 \subseteq B_1$. Therefore, $Cn(A_2) \subseteq Cn(B_1) = B_1$. From $A_1 \subseteq Cn(A_1) \subseteq Cn(A_2)$, we then get $A_1 \subseteq B_1$, and hence $B_1 \in \mathcal{B}_{A_1}^{M_1}$. By (1), $x \dashv\vdash \wedge M_1(B_1) \dashv\vdash \wedge h(M_1)$. So, $x \in \mathcal{O}_3(N, A_2)$ as required. □

We define $\mathcal{O}_3(N) = \{(A, x) : x \in \mathcal{O}_3(N, A)\}$. Example 1 shows that $\mathcal{O}_3$ does not validate the rule of deontic detachment, and hence does not validate CT.

Example 1 (Deontic detachment). Consider the set of norms $N = \{(\top, a), (a, x)\}$. We have $a \in \mathcal{O}_3(N, \top)$, since $M = \{(\top, a)\}$ is a $\top$-witness for a. We also have $x \in \mathcal{O}_3(N, a)$, since $M = \{(a, x)\}$ is an a-witness for x. But we do not have $x \in \mathcal{O}_3(N, \top)$. This may be verified in two steps. First, you identify all the non-empty subsets M of N that are triggered by the input, $M(Cn(\emptyset)) \neq \emptyset$. Next, you go through the list of all these subsets, and check that, for none of them, the smallest relevant B outputs heads whose conjunction is equivalent to x:

M	B	$M(B)$
$\{(\top, a)\}$	$Cn(a)$	$\{a\}$
$\{(\top, a)(a, x)\}$	$Cn(a, x)$	$\{a, x\}$

We illustrate the account with two examples from the literature.

Example 2 ("Change your mind!"). Hansson [12] gives the following example, with credit to Pörn:

> "Consider the hoary example of the man who ought to go to a meeting on August 5 and who ought to send, on August 2, a note explaining his absence, if and only if he is in fact going to be absent." [12, p.425-6]

The example is structurally identical to the Chisholm example [3]. The norms involved may be rendered as $N = \{(\top, m), (m, \neg s)(\neg m, s)\}$, where m and s are for attending the meeting and sending a note, respectively. Given input $\top$, $m \wedge \neg s$ is outputted, but not $\neg s$. This is as it should be. The obligation of $\neg s$ will not be triggered unless the agent is going to fulfil his primary obligation of m. In the violation context $\neg m$, $m \wedge \neg s$ is still outputted. If not, then the following intuitive deontic reasoning pattern would not be supported:

> "August 2 arrives, and though he is able to attend the meeting, he has no intention of doing so. He argues: 'I ought to change my mind, forbear note-writing, and attend the meeting.... My present fulfillment of this obligation will help make up for my sinfully staying at home on the fifth!'." [12, p. 426]

Example 3 ("Sing and dance!"). Conjunction elimination refers to the move from "it ought to be the case that $x \wedge y$" to "it ought to be the case that x". Consider $N = \{(\top, x \wedge y)\}$. Given input $\top$, $x \wedge y$ is outputted, but not x. Goble [6, p.183-184] and Hansen [9, §6.2], among others, have argued against conjunction elimination. There are cases where the two states of affairs (mentioned in the obligation) are only conjunctively required. If the obligation of x was outputted, then (when assessing how well or badly the agent did) a strange consequence would follow, in the event that the agent made x, but not y, true. One would have to acknowledge that (to quote Goble) "he's not a complete scoundrel" [6, p.183], since at least one obligation (albeit a derived one) was fulfilled. Intuitively, one would like to be able to say that *no* obligations have been fulfilled, and that *nothing* right has happened. This may be illustrated with the sing-and-dance example, due to Goble. By making only one conjunct true, the agent makes things worse than if he had done nothing. Suppose there is a party of sing and dance performers given in honour of someone called Gene. Everyone ought to perform a sing and dance routine, because Gene loves them both, and cannot tolerate either without the other. One guest, call him Fred, chooses not to sing but only to dance. Gene is appalled. The party is ruined, because of Gene's tantrum.

Definition 6 (Proof system). $(a, x) \in \mathcal{D}_3(N)$ *if and only if there is a derivation of* (a, x) *from* N *using the rules* $\{$*SI, EQ, ACT*$\}$.

$$ACT\,\frac{(a, x) \qquad (a \wedge x, y)}{(a, x \wedge y)}$$

AND is derivable from SI and ACT. We define $(A, x) \in \mathcal{D}_3(N)$ and $\mathcal{D}_3(N, A)$ as we did for $\mathcal{D}_1$.

Theorem 12. *$\mathcal{O}_3$ validates the rules of $\mathcal{D}_3$ (for individual formulae a).*

Proof. The argument for SI is virtually the same as in the proof of Theorem 11. The argument for EQ is straightforward, and is omitted. We show ACT. Assume that $x \in \mathcal{O}_3(N, a)$, $y \in \mathcal{O}_3(N, a \wedge x)$ and $x \wedge y \notin \mathcal{O}_3(N, a)$. From the first two, it follows that there are finite $M_1, M_2 \subseteq N$ such that $M_1(Cn(a)) \neq \emptyset$, $M_2(Cn(a, x)) \neq \emptyset$, and

$$x \dashv\vdash \wedge M_1(B) \text{ for all } B \in \mathcal{B}_a^{M_1} \tag{2}$$

$$y \dashv\vdash \wedge M_2(B) \text{ for all } B \in \mathcal{B}_{a \wedge x}^{M_2} \tag{3}$$

By Theorem 10,

$$x \dashv\vdash \wedge h(M_1) \tag{4}$$

$$y \dashv\vdash \wedge h(M_2) \tag{5}$$

Therefore,

$$x \wedge y \dashv\vdash \wedge h(M_1) \wedge (\wedge h(M_2)) \tag{6}$$

$$\dashv\vdash \wedge h(M_3) \tag{7}$$

where $M_3 = M_1 \cup M_2$. From the third opening assumption, since $M_3(Cn(a)) \neq \emptyset$, it follows that there is some $B_1 \in \mathcal{B}_a^{M_3}$ such that

$$\text{not-}(\ x \wedge y \dashv\vdash \wedge M_3(B_1)\ ) \tag{8}$$

We have $M_1(B_1) \subseteq M_3(B_1)$, and so $B_1 \in \mathcal{B}_a^{M_1}$. Therefore $x \in B_1$, and hence $a \wedge x \in B_1$. So $B_1 \in \mathcal{B}_{a \wedge x}^{M_2}$ too, since $M_2(B_1) \subseteq M_3(B_1)$. Now,

$$M_3(B_1) = M_1(B_1) \cup M_2(B_1)$$

where $\wedge M_1(B_1) \dashv\vdash x$ and $\wedge M_2(B_1) \dashv\vdash y$. Thus, $\wedge M_3(B_1) \dashv\vdash x \wedge y$, a contradiction. □

Theorem 13 (Soundness). $\mathcal{D}_3(N, A) \subseteq \mathcal{O}_3(N, A)$

Proof. Same argument as for Theorem 3 using Theorems 11 and 12. □

Theorem 14 (Completeness). $\mathcal{O}_3(N, A) \subseteq \mathcal{D}_3(N, A)$

Proof. We give an outline of the proof for the particular case where A is a singleton set $\{a\}$. Suppose that $x \in \mathcal{O}_3(N, a)$. To show: $x \in \mathcal{D}_3(N, a)$. From the former, there is some finite $M \subseteq N$ such that $M(Cn(a)) \neq \emptyset$ and, for all $B \in \mathcal{B}_a^M$, $x \dashv\vdash \wedge M(B)$.

Put $B_1 = Cn(\{a\} \cup \mathcal{D}_3(M, a))$. We have $a \in B_1 = Cn(B_1)$. We also have $M(B_1) \neq \emptyset$, because $Cn(a) \subseteq B_1$. A phasing result from [14] allows, then, to establish that $M(B_1) \subseteq B_1$, so that $B_1 \in \mathcal{B}_a^M$. The opening assumption, then, yields, $x \dashv\vdash \wedge M(B_1)$.

Based on this, one gets a derivation of (a, x) from N as follows. First, note that $M(B_1) \neq \emptyset$. By Definition 1, one gets $x \in \mathcal{O}_1(N, \{a\} \cup \mathcal{D}_3(M, a))$. By Theorem 4, $x \in \mathcal{D}_1(N, \{a\} \cup \mathcal{D}_3(M, a))$, and thus $x \in \mathcal{D}_3(N, \{a\} \cup \mathcal{D}_3(M, a))$. This means that $x \in \mathcal{D}_3(N, \{a, a_1, ..., a_n\})$, where, for each a_i, $a_i \in \mathcal{D}_3(M, a)$. By AND, $\wedge_{i=1}^n a_i \in \mathcal{D}_3(M, a)$. Since $M \subseteq N$, $\wedge_{i=1}^n a_i \in \mathcal{D}_3(N, a)$. A derivation of (a, x) from N is shown below.

$$
\dfrac{\text{ACT}\,\dfrac{(a, \wedge_{i=1}^n a_i) \qquad (a \wedge (\wedge_{i=1}^n a_i), x)}{(a, \wedge_{i=1}^n a_i \wedge x)} \qquad \dfrac{x \vdash \wedge_{i=1}^n a_i}{\wedge_{i=1}^n a_i \wedge x \dashv\vdash x}}{(a, x)}\ \text{EQ}
$$

The argument for $x \vdash \wedge_{i=1}^n a_i$ appeals to two lemmas:

- $x \dashv\vdash \wedge h(M)$, Theorem 10
- $h(M) \vdash a_i$, for all $1 \leq i \leq n$ – the proof of this is by induction on the length of the derivation of (a, a_i)

The argument may be generalized to an input set A of arbitrary cardinality. □

5 Properties

In a companion paper [17], we identify some desirable properties, which are all satisfied by $\mathcal{O}_3$. These are listed in Table 1. We refer the reader to the aforementioned paper for the motivation and a discussion of these properties. These also hold for $\mathcal{O}_1$ and $\mathcal{O}_2$, when replacing out_3 by out_1 or out_2, respectively. On the left hand side of the table, exact factual detachment (EFD) and violation detection (VD) characterise what is special about *deontic* logic, while substitution (SUB), replacements of logical equivalents (RLE), implication (IMP) and paraconsistency (PC) say something about *logic*. We use the notation $x[\sigma]$ to denote a substitution instance of x. Thus, $x[\sigma]$ is obtained from x by replacing uniformly, in x, all occurrences of a propositional letter by the same propositional formula. $A[\sigma]$ and $N[\sigma]$ extend the notion of substitution instance to sets of formulae, and sets of norms in the straightforward way. We write $N \approx M$ whenever M is obtained from N, by replacing each $(b, y) \in N$ with some (c, z) such that b is equivalent with c, and y is equivalent with z. Implication makes use of the so-called materialisation $m(N)$ of a normative system N, which means that each norm (a, x) is interpreted as a material conditional $a \to x$, i.e. as the propositional sentence $\neg a \vee x$. We distinguish between violations $V(N, A) = \{x \in \mathcal{O}_3(N, A) \mid \neg x \in Cn(A)\}$ and non-violations (or cues for action) $\overline{V}(N, A) = \mathcal{O}_3(N, A) \setminus V(N, A)$.

On the right hand side of the table, norm monotony (NM) and norm induction (NI) are called "norm change properties", because the normative system N is no longer held constant. Together, exact factual detachment, norm monotony and norm induction are equivalent to saying that $\mathcal{O}_3(N)$ is a closure operator.

Table 1. Properties [17]

EFD	$(x, y) \in N \Rightarrow y \in \mathcal{O}_3(N, x)$	NM	$\mathcal{O}_3(N) \subseteq \mathcal{O}_3(N \cup M)$
VD	$(A, y) \in \mathcal{O}_3(N) \Rightarrow (A \cup \{\neg y\}, y) \in \mathcal{O}_3(N)$	NI	$M \subseteq \mathcal{O}_3(N) \Rightarrow \mathcal{O}_3(N) = \mathcal{O}_3(N \cup M)$
SUB	$x \in \mathcal{O}_3(N, A) \Rightarrow x[\sigma] \in \mathcal{O}_3(N[\sigma], A[\sigma])$		
RLE	$N \approx M \Rightarrow \mathcal{O}_3(N) \subseteq \mathcal{O}_3(M)$	IO	$\mathcal{O}_3(N, A) \subseteq out_3(N, A)$
IMP	$\mathcal{O}_3(N, A) \subseteq Cn(m(N) \cup A)$	R	$outf_3(N, A) = outf_3(\mathcal{O}_3(N), A)$
PC	$x \in \overline{V}(N, A) \Rightarrow \exists M \subseteq N : x \in \mathcal{O}_3(M, A)$ and $\mathcal{O}_3(M, A) \cup A$ consistent	SR	$outf_3(N \cup M, A) = outf_3(\mathcal{O}_3(N) \cup M, A)$

Finally, the reusability properties relate the system to standard I/O logic: inclusion in reusable output (IO), redundancy (R) and strong redundancy (SR). Their formulation appeals to some key notions of so-called constrained input/output logic, developed by Makinson and van der Torre [15] in order to reason about norm violation:

$$conf(N, A) = \{N' \subseteq N \mid out(N', A) \cup A \text{ consistent}\}$$
$$maxf(N, A) = \{N' \in conf(N, A) \mid N' \subseteq \text{-maximal}\}$$
$$outf(N, A) = \{out(N.A) \mid N' \in maxf(N, A)\}$$

It is worth recalling the reason why consistency checks were introduced in I/O logic. This was done in relation to contrary-to-duty reasoning. In unconstrained input/output logic, a violation leads to outputting the whole propositional language. This deontic explosion is not a property of the logics we introduce in this paper, as a direct consequence of the lack of the weakening rule. We believe that the unconstrained logics introduced in this paper can capture some aspects of contrary-to-duty-reasoning.

There is another property that acts as a bridge between the logics defined in this paper and the traditional input/output logics. It was not listed in [17] because it may not necessarily be considered a desirable property. This is the property: $out_1(N, A) = Cn(\mathcal{O}_1(N, A))$ and $out_3(N, A) = Cn(\mathcal{O}_3(N, A))$. Somewhat surprisingly, we do not have in general $out_2(N, A) = Cn(\mathcal{O}_2(N, A))$. For a counter-example, take $N = \{(a, x), (b, x \wedge y)\}$ and $A = \{a \vee b\}$. We leave it for future research to define a logic $\mathcal{O}'_2$ satisfying not only the properties in Table 1, but also the requirement $out_2(N, A) = Cn(\mathcal{O}'_2(N, A))$.

6 The Way Forward

This paper has extended Stolpe's results on I/O logics without weakening in two directions. First, we have shown how to account for reasoning by cases. Second, we have shown how to inject a new ("aggregative") form of cumulative transitivity, which we think is more suitable for normative reasoning. Soundness and completeness theorems for the proposed systems have been reported.

More work is to be carried out. First, it would be interesting to know if the two semantics proposed here may be merged to yield a new basic reusable operation

out_4, with ACT, but not WO, amongst its primitive rules. Second, we have found that ACT has two drawbacks. The first one is that ACT derives the so-called pragmatic oddity [18]. The second one is that, in a violation context, ACT creates 'irrelevant' obligations, and thus the account faces an over-generation problem: more obligations are generated than it seems right. Let us call this the irrelevant obligation problem. The derivation to the left illustrates the pragmatic oddity with the dog-and-sign scenario—the letters d and s are for "there is a dog" and "there is a warning sign," respectively. The derivation to the right illustrates the irrelevant obligation problem.

$$\text{ACT}\ \dfrac{\text{SI}\ \dfrac{(\top, \neg d)}{(d, \neg d)} \qquad \text{SI}\ \dfrac{(d, s)}{(d \wedge \neg d, s)}}{(d, \neg d \wedge s)} \qquad\qquad \text{ACT}\ \dfrac{\text{SI}\ \dfrac{(\top, \neg x)}{(x, \neg x)} \qquad \text{SI}\ \dfrac{(b, y)}{(x \wedge \neg x, y)}}{(x, \neg x \wedge y)}$$

Pragmatic oddity — Irrelevant obligation

There are similarities between the two problems. However, we feel that the two should be distinguished. While it is clear that the derivation to the right should always be blocked, it is less clear whether the one to the left should always be blocked too. Indeed, one can think of examples in which the pragmatic oddity does make sense. For instance, if you do not pay the tax you own, you usually have to pay both a fine and your tax. Furthermore, as we will see in a moment, a solution to the irrelevant obligation problem may not be a solution to the pragmatic oddity.

This irrelevant obligation problem was pointed out by Stolpe [21, p. 134], in relation to MCT/CT. His diagnosis is that plain transitivity is more suitable for normative reasoning than cumulative transitivity. Plain transitivity is the rule "From (a, x) and (x, y), infer (a, y)". We will not follow up on his suggestion: the counter-examples alluded to in the introduction discredit both forms of transitivity. We are presently studying other ways around. A number of solutions naturally come into mind. These are listed below.

One first obvious possibility is to restrict the application of ACT, allowing it to be applied only if, e.g., the output is consistent with the input. This would solve both problems. This solution is proof-theoretical in nature. It would remaim to see how to build it in the semantics.

A second possibility is to adopt a more procedural approach, by incorporating 'backtesting' into the account:

Backtesting

$$(A, x) \in \mathcal{O}'_3(N) \text{ iff } \exists A' \subseteq Cn(A) \text{ with } (A', x) \in \mathcal{O}_3(N) \text{ and } A' \cup \{x\} \nvdash \bot$$

Intuitively, the definition says: for x to be obligatory in context A, it must have been the case that x was obligatory before the violation occurred, viz in context $A' \subseteq Cn(A)$ with A' consistent with x. Thus, obligations do not 'drown' in a violation context. We leave it to the reader to verify that backtesting filters out pragmatic oddities.

A third option is to change the base logic from classical logic to some suitable sub-classical logic. In order to resolve the irrelevant obligation problem, any logic that rejects the principle *ex falso quodlibet*, $\{x, \neg x\} \vdash y$, will do. A number of paraconsistent logics are available (for an overview, see [19]). Devised by Dosen[4], the so-called system **N** is amongst the simplest ones. It may fruitfully be used to illustrate the latter point. System **N** comes with a Kripke-type possible worlds semantics similar to that used for intuitionistic logic. The main difference is that the evaluation rules for $\rightarrow$ and $\neg$ use separate accessibility relations. The system is strictly included in Johansson's well-known system for minimal negation. One key difference is that, unlike the latter, the former does not keep *ex falso* in the following modified form: $\{x, \neg x\} \vdash \neg y$. The fact that such a system would do the job can easily be checked. Put $N = \{(\top, \neg x), (b, y)\}$ and $A = \{x\}$. We have

	M	B	$M(B)$
1.	$\{(\top, \neg x)\}$	$Cn(x, \neg x)$	$\{\neg x\}$
2.	$\{(\top, \neg x)(b, y)\}$	$Cn(x, \neg x)$	$\{\neg x\}$

The bottom line is this. System **N** keeps the principle *verum ex quodlibet*. This is the law $\Gamma \vdash \top$, where Γ is a set of formulae. So, on line 2, $\top \in B = Cn(x, \neg x)$, and thus $\neg x \in M(B)$. But $y \notin M(B)$ because, in the absence of *ex falso*, $b \notin B = Cn(x, \neg x)$. We use system **N** for illustrative purposes only. It could be that a more sophisticated paraconsistent logic is needed. Futhermore, to handle the pragmatic oddity, we need to do more than just let *ex falso* go away.

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