Input/output logic

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October 26, 2015

1 Introduction

This handout deals with the input/output (I/O) logic developed by Makinson and van der Torre [2, 3, 4]. The basic idea is to extend the theory of conditional norms from modal logic to the abstract study of conditional codes viewed as sets of relations between Boolean formulas.

The semantics is operational rather than truth-functional. The meaning of the deontic concepts is given in terms of a set of procedures yielding outputs for inputs. The basic mechanism underpining these procedures is that of detachment. The associated proof-theory is formulated in terms of derivation rules operating on pairs of formulas rather than individual formulas.

This handout focuses on the I/O analysis of the concept of obligation as described in [2]. This is the simplest case; no extra machinery is used to filter out the output. What is delivered is (as Makinson [1] calls it) the "gross output".

2 Preliminaries

Let $\mathbb{L}$ be the set of all the formulas of classical propositional logic. A normative system N is set of (ordered) pairs (a, x) of formulas. Intuitively, a pair (a, x) denotes a conditional obligation. (a, x) is read as "given a, it ought to be that x". a is called the body (or antecedent), and x the head (or consequent). $(\top, x)$ denotes the unconditional obligation of x, where $\top$ is a tautology.

Given a set A of formulas (input set), we use the notation $out(N, A)$ to denote the output of A under N. We also use the equivalent notation $(A, x) \in out(N)$. When A is a singleton set, curly brackets are omitted.

⚠ Do not confuse $x \in out(N, a)$ with $(a, x) \in N$.

Definition 1 (Consequence)**.** *$Cn(A)$ denotes the set of logical consequences of A in classical propositional logic. That is, $Cn(A) = \{x : A \vdash x\}$ (for $\vdash$, read 'proves').*

Recall from the *Bridges* handout:

Theorem 1. *Cn is a Tarskian operation, viz it satisfies the properties:*

$$A \subseteq Cn(A) \qquad \text{(Inclusion)}$$
$$A \subseteq B \Rightarrow Cn(A) \subseteq Cn(B) \qquad \text{(Monotony)}$$
$$Cn(A) = CnCn(A) \qquad \text{(Idempotence)}$$

Remark 1. Cn satisfies the property of compactness:

$$x \in Cn(A) \Rightarrow \exists \text{ finite } A' \subseteq A \mid x \in Cn(A')$$

Definition 2 (Image)**.** *$N(A) = \{x : (a, x) \in N$ for some $a \in A\}$. For $N(A)$, read "the N of A".*

Example 1.

N	A	$N(A)$
$\{(a_1, x_1), (a_2, x_2)\}$	$\{a_1\}$	?
$\{(a_1, x_1), (a_2, x_2)\}$	$\{a_1, x_2\}$	?
$\{(a_1, x_1), (a_2, x_2)\}$	$\{a_1, a_2\}$	?
$\{(a_1, x_1), (a_2, x_2)\}$	$\emptyset$	?

Theorem 2 (Monotony)**.** $A \subseteq B \Rightarrow N(A) \subseteq N(B)$.

Proof. In class. □

EXERCISES

Exercise 2.1. Express the set of all the tautologies in the Cn notation.

Exercise 2.2. Let $N = \{(a, x), (x, y), (x \wedge y, z)\}$. Calculate $N(N(\{a, x, y\}))$.

Exercise 2.3. What is $N(\mathbb{L})$?

3 Simple-minded (out_1)

The operation out_1 (called "simple-minded") spells out the basic mechanism underpinning the operation of detachment in the non-iterated case.

Definition 3. *We define $out_1(N, A)$ as*

Cn(N(Cn(A)))

$$out_1(N) = \{(A, x) : x \in out_1(N, A)\}.$$

Example 2. *Let $N = \{(a, x), (a \vee b, y)\}$. Put $A = \{a\}$.*

A	$Cn(A)$	$N(Cn(A))$	out_1
a	$Cn(a)$	$\{x, y\}$	$Cn(x, y)$

Remark 1. *$x \in out_1(N, A)$ says: $A \vdash \wedge_{i=0}^{n} a_i$ and $\wedge_{i=0}^{n} x_i \vdash x$, where (a_1, x_1), ..., $(a_n, x_n) \in N$.*

Definition 4. $(a, x) \in deriv_1(N)$ *("(a, x) is derivable from N) whenever (a, x) is in the least superset of N that includes $(\top, \top)$ and is closed under the rules $\{$SI,AND,WO$\}$.*

$$\text{SI} \; \frac{(a, x) \qquad b \vdash a}{(b, x)}$$

$$\text{WO} \; \frac{(a, x) \qquad x \vdash y}{(a, y)}$$

$$\text{AND} \; \frac{(a, x) \qquad (a, y)}{(a, x \wedge y)}$$

SI and WO abbreviate "strengthening of the input" and "weakening of the output", respectively.

Given a set A of formulas, $(A, x) \in deriv_1(N)$ whenever $(a, x) \in deriv_1(N)$ for some conjunction a of formulas in A.

Put $deriv_1(N, A) = \{x : (A, x) \in deriv_1(N)\}$.

Example 3. *Let $N = \{(a \vee b, x)\}$. We have $(b, x \vee y) \in deriv_1(N)$. Indeed:*

1.	$(a \vee b, x)$	*Assumption*
2.	(b, x)	*1, SI*
3.	$(b, x \vee y)$	*2, WO*

Theorem 3. *out_1 validates $\{$SI, AND, WO$\}$ (for input a).*

⚠ That e.g. SI is validated means that $x \in out_1(N, b)$ whenever $x \in out_1(N, a)$ and $b \vdash a$.

Proof. In class. □

Corollary 1 (Soundness)**.** *$deriv_1(N) \subseteq out_1(N)$.*

Theorem 4 (Completeness)**.** *$out_1(N) \subseteq deriv_1(N)$.*

Proof. In class. □

EXERCISES

Exercise 3.1. Show that OR fails for out_1.

Exercise 3.2. Let $N = \{(a, x), (b, y), (a \vee b, z)\}$. What is $out_1(N, \{a, b\})$? Let $N = \{(\top, x)\}$. What is $out_1(N, a)$?

Exercise 3.3. Show that AND is valid. Show that CONT ("contraposition") is not valid.

$$\text{CONT} \; \frac{(a, x)}{(\neg x, \neg a)}$$

Exercise 3.4. For $N = \{(\top, x), (a, y \wedge z)\}$, do we have $(a, x \wedge z) \in deriv_1(N)$?

4 Basic (out_2)

The operation out_2 (called "basic") injects (OR) into out_1 so that reasoning by cases is supported. Throughout this section, V is a set of formulas. We say that V extends V' if $V' \subseteq V$, and that V is a proper extension of V' if $V' \subset V$.

Definition 5. *V is maximal consistent if*

$$V \nvdash \bot, \text{ and} \tag{1}$$

$$y \notin V \Rightarrow V \cup \{y\} \vdash \bot \tag{2}$$

Intuitively: V is consistent, and no proper extension of V is consistent.

For some notation, MCS and MCE abbreviate "maximal consistent set", and "maximal consistent extension", respectively.

Fact 1. *If V is a MSC, then*

$$Cn(V) = V \tag{3}$$

$$\text{either } b \in V \text{ or } \neg b \in V \tag{4}$$

$$\text{if } b \vee c \in V \text{ then: } b \in V \text{ or } c \in V \tag{5}$$

(3) says that V is closed under Cn. (4) and (5) are called "$\neg$-completeness" and "saturatedness" (or "primeness"), respectively.

V is called complete if V is a MCS or equal to $\mathbb{L}$.

Definition 6. *We define $out_2(N, A)$ as*

$$\boxed{\cap\{Cn(N(V)) : A \subseteq V, V \text{ complete }\}}$$

There is always at least one complete V extending A, namely $\mathbb{L}$ itself.

Put $out_2(N) = \{(A, x) : x \in out_2(N, A)\}$.

Example 4. *Let $N = \{(a, x), (b, x)\}$ and $A = \{a \vee b\}$.*

V	$N(V)$	$Cn(N(V))$
$\mathbb{L}$	$\{x\}$	$Cn(x)$
MCE of $\{a \vee b\}$	$\{x\}$	$Cn(x)$

↳ *$out_2(N, A) = Cn(x)$.*

Remark 2. *Because $V = Cn(V)$, out_2 may be rephrased thus:*

$$out_2(N, A) = \cap\{out_1(N, V) : A \subseteq V, V \text{ complete}\}$$

Definition 7. *$(a, x) \in deriv_2(N)$ whenever (a, x) is in the least superset of N that includes $(\top, \top)$ and is closed under the rules $\{$SI, AND, WO, OR$\}$, where OR is*

$$\text{OR} \; \frac{(a, x) \qquad (b, x)}{(a \vee b, x)}$$

Put $deriv_2(N, A) = \{x : (A, x) \in deriv_2(N, A)\}$.

Theorem 5. *out_2 validates {SI, WO, AND, OR} (for input a).*

Proof. We only show OR. Let $x \in out_2(N, a)$ and $x \in out_2(N, b)$. To show: $x \in out_2(N, a \vee b)$.

Let V be a complete set extending $a \vee b$. By definition of a complete set, i) either $a \in V$ or ii) $b \in V$. In case i), the assumption $x \in out_2(N, a)$ allows us to conclude $x \in Cn(N(V))$. In case ii), the assumption $x \in out_2(N, b)$ allows us to conclude $x \in Cn(N(V))$. Either way, $x \in Cn(N(V))$, which suffices for $x \in out_2(N, a \vee b)$. □

Corollary 2 (Soundness)**.** $deriv_2(N) \subseteq out_2(N)$.

Theorem 6 (Completeness)**.** $out_2(N) \subseteq deriv_2(N)$.

Proof. See [2, Obs. 2]. □

EXERCISES

Exercise 4.1. Show out_2 validates the rules of $deriv_1$.

Exercise 4.2. Let $N = \{(a, x), (b, y)\}$. Show that $(a \vee b, x \vee y) \in deriv_2(N)$.

5 Reusable (out_3)

The operation out_3 (called "reusable output") generalizes out_1 to the iterated case.

Definition 8. *We define $out_3(N, A)$ as*

$$\cap\{Cn(N(B)) : A \subseteq B = Cn(B) \supseteq N(B)\}$$

There is always at least one such B, namely $\mathbb{L}$.

Define $out_3(N) = \{(A, x) : x \in out_3(N, A)\}$.

Example 5. *Let $N = \{(a, x), (a \wedge x, y)\}$ and $A = \{a\}$. We have:*

B	*N(B)*	$Cn(N(B))$
$Cn(a, x, y)$	$\{x, y\}$	$Cn(x, y)$

↳ *$out_3(N, A) = Cn(x, y)$.*

Below: an inductive characterization of out_3.

Definition 9 (Bulk increments, [5])**.** *Define $out_3^b(N, A) = \cup_{i=0}^{\omega} A_i$ where*

$$A_0 = out_1(N, A)$$
$$A_{i+1} = Cn(A_i \cup out_1(N, A_i \cup A))$$

Intuitively: output is continuously recycled as input, detaching heads of rules, whenever possible.

All the A_i's are linearly ordered under $\subseteq$.

Example 6. *In Exa. 5, we have*

$$\begin{aligned} A_0 &= Cn(x) \\ A_1 &= Cn(Cn(x) \cup out_1(N, Cn(x) \cup \{a\})) \\ &= Cn(Cn(x) \cup Cn(x, y)) \\ &= Cn(x, y) \\ A_2 &= A_1 \\ &\vdots \end{aligned}$$

Theorem 7. *out_3 and out_3^b are equivalent.*

Proof. This is [5, Th. 4.3.12 and Th. 4.3.13]. □

Definition 10. *$(a, x) \in deriv_3(N)$ whenever (a, x) is in the least superset of N that includes $(\top, \top)$ and is closed under the rules {SI, AND, WO, CT}, where CT abbreviates "Cumulative Transitivity". This is the rule*

$$CT\ \frac{(a, x) \qquad (a \wedge x, y)}{(a, y)}$$

Put $deriv_3(N, A) = \{x : (A, x) \in deriv_3(N, A)\}$.

Remark 3. *Plain transitivity is a derived rule.*

$$CT\ \frac{(a, x) \qquad \dfrac{(x, y)}{(a \wedge x, y)}\ SI}{(a, y)}$$

Theorem 8. *out_3 validates {SI, AND, WO, CT} (for input a).*

Proof. We only show CT. Let $x \in out_3(N, a)$ and $y \in out_3(N, a \wedge x)$. To show: $y \in out_3(N, a)$.

Let $B \mid a \in B = Cn(B) \supseteq N(B)$. The assumption $x \in out_3(N, a)$ yields $x \in Cn(N(B))$. But $N(B) \subseteq B = Cn(B)$. By monotony for Cn, $Cn(N(B)) \subseteq Cn(B)$, so that $x \in Cn(B)$, hence $x \in B$, from which one gets $a \wedge x \in B$. The assumption $y \in out_3(N, a \wedge x)$ now yields $y \in Cn(N(B))$, which suffices for $y \in out_3(N, a)$. □

Corollary 3 (Soundness)**.** $deriv_3(N) \subseteq out_3(N)$.

Theorem 9 (Completeness)**.** $out_3(N) \subseteq deriv_3(N)$.

Proof. See [2, Obs. 6]. □

EXERCISES

Exercise 5.1. Show that out_3 validates SI.

6 Basic reusable (out_4)

The operation out_4 (called "basic reusable") injects (OR) into out_3 so that reasoning by cases is supported.

Definition 11. *We define $out_4(N, A)$ as*

$$\cap\{Cn(N(V)) : A \subseteq V \supseteq N(V), V \text{ complete}\}$$

Define $out_4(N) = \{(A, x) : x \in out_4(N, A)\}$.

Definition 12. $(a, x) \in deriv_4(N)$ *whenever* (a, x) *is in the least superset of* N *that includes* $(\top, \top)$ *and is closed under the rules* $\{$*SI, AND, WO, CT, OR*$\}$.

Put $deriv_4(N, A) = \{x : (A, x) \in deriv_4(N, A)\}$.

Theorem 10. out_4 *validates* $\{$*SI, AND, WO, CT, OR*$\}$ *(for input a)*

Corollary 4 (Soundness)**.** $deriv_4(N) \subseteq out_4(N)$.

Theorem 11 (Completeness)**.** $out_4(N) \subseteq deriv_4(N)$.

Proof. See [2, Obs. 11]. □

7 Summary

Table 1 shows the I/O operations and the associated rules.

Table 1: I/O systems

Output operation	Rules
Simple-minded (out_1)	{SI, WO, AND}
Basic (out_2)	{SI, WO, AND}+{OR}
Reusable s-m (out_3)	{SI, WO, AND}+{CT}
Reusable basic (out_4)	{SI, WO, AND}+{OR,CT}

References

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[2] D. Makinson and L. van der Torre. Input-output logics. *Journal of Philosophical Logic*, 29:383–408, 2000.

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[5] A. Stolpe. *Norms and Norm-System Dynamics*. PhD thesis, Department of Philosophy, University of Bergen, Norway, 2008.