I/O logics with a consistency check

Xavier Parent
University of Luxembourg
xavier.parent@uni.lu

Leendert van der Torre
University of Luxembourg
leon.vandertorre@uni.lu

Abstract

Norm-based semantics to deontic logic typically come in an unconstrained and constrained version, where the unconstrained version comes with a proof system, and the constraints handle phenomena such as dilemmas, contrary-to-duty reasoning, uncertainty and defeasibility. This is analogous to the use of rule-based languages in non-monotonic logic such as logic programming or default logic, but in contrast to the traditional modal framework. Traditionally, for example, specific modal deontic logics have been defined that make dilemmas inconsistent, as well as other modal deontic logics representing dilemmas in a consistent way. This issue was raised recently in the input/output logic framework, and weaker unconstrained logics have been defined handling phenomena like dilemmas and contrary-to-duty reasoning. In this paper we introduce a semantics and proof theory for a system with various desirable properties. We show that our new deontic logic satisfies a criterion posed several years ago by Broersen and van der Torre, allowing deontic detachment while preventing Prakken and Sergot's pragmatic oddities as well as Sergot's drowning problem.

Keywords: Deontic logic, normative systems, I/O logic, reusable output, consistency check

Thanks to three anonymous reviewers for valuable comments. This work is supported by the European Union's Horizon 2020 research and innovation programme under the Marie Curie grant agreement No: 690974 (Mining and Reasoning with Legal Texts, MIREL).

1 Introduction

This paper deals with I/O logic, initially devised by Makinson and van der Torre [7]. It falls within the category of what Hansen [6] calls a "norm-based" deontic logic. The central question of a norm-based deontic logic is : what obligations can be detached from a set N of (explicitly given) rules or conditional norms in a given context? The approach is in this regard very different from the more traditional one, aiming at identifying a set of "logical laws" using a possible worlds semantics.

1.1 Aim of paper

We first explain the main purpose of this paper, and its contribution to the literature on I/O logic. An overview of the I/O systems that have been studied so far in the literature is shown in Table 1 along with the system studied in this paper. A pair (a, x) denotes a conditional obligation. (a, x) is read: "if a, then x is obligatory". The columns show the rules characterising a system. The symbol "+" indicates the presence of a rule, and the symbol "-" its abscence. The right-most column gives the paper(s) where the system has been studied. Each system comes with a semantics, and a completeness theorem linking the semantics with the proof theory. The last system mentioned in the table is the strongest one. It collapses with the system of classical propositional logic: (a, x) is derivable from N if and only if $m(N) \vdash a \to x$, where $\vdash$ is the deducibility relation used in classical propositional logic and $m(N)$ is the set of materializations of N, viz $\{b \to y : (b, y) \in N\}$. We list the rules in the order in which they appear in the table, starting from the left-most column.

- EQ (equivalence of the output): from (a, x), $x \vdash y$ and $y \vdash x$, infer (a, y)
- SI (strengthening of the input): from (a, x) and $b \vdash a$, infer (b, x) [1]
- WO (weakening of the output): from (a, x) and $x \vdash y$, infer (a, y)
- R-AND (restricted AND): from (a, x), (a, y) and $a \wedge x \wedge y$ consistent, infer $(a, x \wedge y)$
- R-AND': from (a, x), (a, y), $a \wedge x$ consistent and $a \wedge y$ consistent, infer $(a, x \wedge y)$
- AND: from (a, x) and (a, y), infer $(a, x \wedge y)$
- OR: from (a, x) and (b, x), infer $(a \vee b, x)$
- R-ACT (restricted ACT'): from (a, x), $(a \wedge x, y)$ and $a \wedge x \wedge y$ consistent, infer $(a, x \wedge y)$
- ACT (aggregative CT): from (a, x), $(a \wedge x, y)$, infer $(a, x \wedge y)$
- MCT (mediated CT): from (a, x'), $(a \wedge x, y)$, $x' \vdash x$, infer (a, y)
- CT (cumulative transitivity): from (a, x) and $(a \wedge x, y)$, infer $(a, x \wedge y)$

[1] Given SI, the analog of EQ holds for the input.

- ID (identity): (a, a)

In this paper, we want to investigate the effects of adding R-ACT to {SI,EQ}. (Given SI, R-ACT entails R-AND.) Our aim is to define an I/O operation validating the triplet of rules {SI, R-ACT, EQ}, and to establish a completeness theorem showing the equivalence between the semantics and the proof theory. The I/O operation is called reusable, because the output can be recycled as input (under suitable conditions). The system characterised by {SI, R-ACT, EQ} is weaker than the one characterised by {SI, ACT, EQ}, but stronger than the one characterised by {SI, R-AND, EQ}. In the presence of SI, R-ACT implies R-AND, but not conversely.

EQ	SI	WO	R-AND	R-AND'	AND	OR	R-ACT	ACT	MCT	CT	ID	References
+	+	-	+	-	-	-	-	-	-	-	-	[14]
+	+	-	+	-	-	-	+	-	-	-	-	this paper
+	+	-	+	+	-	-	-	-	-	-	-	[14]
+	+	-	+	+	+	-	-	-	-	-	-	[18, 17, 12]
+	+	-	+	+	+	-	+	+	-	-	-	[12]
+	+	-	+	+	+	-	+	+	+	+	-	[18, 17]
+	+	-	+	+	+	+	-	-	-	-	-	[12]
+	+	+	+	+	+	-	-	-	-	-	-	[7]
+	+	+	+	+	+	-	-	-	-	-	+	[7]
+	+	+	+	+	+	-	+	+	+	+	-	[7, 11]
+	+	+	+	+	+	-	+	+	+	+	+	[7]
+	+	+	+	+	+	+	-	-	-	-	-	[7, 11, 19]
+	+	+	+	+	+	+	-	-	-	-	+	[7]
+	+	+	+	+	+	+	+	+	+	+	-	[7]
+	+	+	+	+	+	+	+	+	+	+	+	[7]

Table 1: Overview of I/O systems

1.2 Broersen and van der Torre's open problem

In order to motivate this work, we explain how the pair of rules {SI, R-ACT} handles a problem pointed out by Broersen and van der Torre in their survey of open problems in deontic logic [2]. For each problem, they discuss traditional and new research questions. The one we will focus on is related to the topic of contrary-to-duty reasoning. After having introduced the traditional problems surrounding this topic, and identified a few new questions, they make the following observation:

> "The pragmatic oddity is the derivation of the conjunction 'you should keep your promise and apologize for not keeping it' from 'you should keep your promise', 'if you do not keep your promise you should apologize' and 'you do not keep your promise' [15]. Note that the sentences of this problem have the same structure as those of the Chisholm scenario. The drowning problem [also called by Parent and van der Torre [13] the violation detection problem][2] is that many solutions of the pragmatic oddity cancel the obligation in case of violation, such that for violations $\neg p \wedge \bigcirc p$, the violated obligation $\bigcirc p$ is no longer derivable." [2, p. 64]

They go on to ask the following question:

> "New question 11: how to prevent the pragmatic oddity without creating the drowning problem?"

Example 1 shows how the proposed system handles this problem. To keep things simple, we make our point using R-AND, which is derivable from the pair of rules {SI,R-ACT}.

Example 1 (Broken promise)**.** *Let* k *and* a *stand for* keeping one's promise *and for* apologizing*, respectively. Consider the following derivation, in which a blocked derivation step is represented by a dashed line.*

$$\cfrac{SI\,\dfrac{(\top, k)}{(\neg k, k)} \qquad (\neg k, a)}{(\neg k, k \wedge a)}\ \textit{R-AND}$$

On the one hand, the drowning problem (or the violation detection problem) does not occur, because SI allows us to move from $(\top, k)$ *to* $(\neg k, k)$*. Constrained I/O logic [8] blocks such a move:* k *is not consistent with* $\neg k$*. On the other hand, the pragmatic oddity is avoided, because R-AND cannot be applied to get* $(\neg k, k \wedge a)$*:* $k \wedge \neg k \wedge a$ *is not consistent.*

[2] The name "drowning problem" was suggested orally by M. Sergot to the first author of the present paper, in relation with the non-monotonic approaches to contrary-to-duty reasoning.

Three remarks are in order:

- The rule R-AND' ("from (a, x), (a, y), $a \wedge x$ consistent and $a \wedge y$ consistent, infer $(a, x \wedge y)$") characterizing the second of the two I/O systems discussed in Parent and van der Torre [14] also blocks the pragmatic oddity.
- This treatment of the pragmatic oddity is very much in the spirit of Prakken and Sergot [16]'s own treatment. They (rightly, in our view) stress that primary and CTD obligations are of a different kind. Our proposal is not to allow them to aggregate using the AND rule, because of this difference in nature. This point was already made by Parent and van der Torre [14].
- R-AND is enough to tackle Broersen and van der Torre's problem. But there is an independent reason for using the stronger rule R-ACT. It is that a system with SI and R-AND only does not allow to chain norms together, and does not support any form of transitivity. This motivates the attempt made in this paper to extend the account described by Parent and van der Torre [14] so it can handle iterations of successive detachments. Since (given SI) ACT implies AND, the combination {ACT, R-AND} must be ruled out. Prima facie, there are other combinations that might be worthwhile studying, like {CT, R-AND}. However, it is still unknown what the corresponding I/O operations would be like. This is why they are not considered in this paper.

The present paper is technical. Our main interest is in formal systems. The proof of completeness we give is not a straightforward adaptation of proofs given elsewhere. As always the devil is in the details. There are two challenging complicating factors that make the proof non-trivial, and worth reporting. First, when calculating the output, one looks at what is "triggered" not by $Cn(A)$, the set of consequences of input set A, but by some $B \subseteq Cn(A)$. This is needed to resolve the violation detection (or drowning) problem: SI is supported. Second, the proposed I/O operations have a built-in "consistency check". They are thus close in spirit to the constrained I/O operations developed by Makinson and van der Torre [8]. The objective is the same: to filter excess output using consistency checks. There are similarities between the two frameworks, but also important differences. First, all the constrained I/O operations face the violation detection problem. Second, in contrast to the constrained I/O operations, the I/O operations studied in this paper have a proof theory. Their built-in consistency check (used to filter excess output) translates into a consistency proviso restraining the application of a rule. This explains the running title of this paper, "Consistent reusability".

This paper follows a straightforward structure. Section 2 gives the required background. Section 3 presents the system, and shows soundness and completeness.

2 Background

In this section we recall some basic definitions and a result from Parent and van der Torre [14], which will be used in the paper.

A normative code is a set N of pairs $(a,\ x)$, where a and x are two formulae of classical propositional logic. Each pair represents a conditional obligation. a is called the body, and x is called the head. Given $M \subseteq N$, $h(M)$ denotes the set of all the heads of the pairs in M, and $b(M)$ denotes the set of all the bodies of the pairs in M. We use the standard notation $(\top, x)$ for the unconditional obligation of x, where $\top$ stands for a tautology like $a \vee \neg a$. $\mathcal{L}$ is the set of all formulae of classical propositional logic. Given an input $A \subseteq \mathcal{L}$, and a normative system N, $N(A)$ denotes the image of N under A, i.e., $N(A) = \{x : (a, x) \in N \text{ for some } a \in A\}$. $Cn(A)$ denotes the set $\{x : A \vdash x\}$, where $\vdash$ is the deducibility relation used in classical propositional logic. The notation $x \dashv\vdash y$ is short for $x \vdash y$ and $y \vdash x$. We use PL as an abbreviation for (classical) propositional logic.

In input/output logic, the main semantical construct has the form: $x \in O(N, A)$. This is read as follows: given input A (state of affairs), x (obligation) is in the output under norms N. The proof-theory is given in terms of inference rules manipulating pairs of Boolean formulas instead of formulas.

Definition 1 reformulates one of the two new I/O operations put forth by Parent and van der Torre [14]–both formulations are equivalent. The I/O operation is written O_1. The definition says the following. Given input A, x is outputted if the following condition holds: x is logically equivalent to the conjunction of all the heads of all the pairs in a non-empty and finite $M \subseteq N$, whose bodies are all in $Cn(A)$, and which are "collectively" consistent with x. Formally:

Definition 1 (Semantics, single-step detachment)**.** $x \in O_1(N, A)$ *iff there is some finite* $M \subseteq N$ *and a set* $B \subseteq Cn(A)$ *such that* $M \neq \emptyset$, $B = b(M)$, $x \dashv\vdash \wedge h(M)$ *and* $\{x\} \cup B$ *is consistent.*

As usual, $O_1(N) = \{(A, x) : x \in O_1(N, A)\}$.

In Parent and van der Torre [14], the account has been applied to a number of benchmark examples from literature. The proposed account has been devised to handle simultaneously the two main categories of benchmark problems discussed in deontic logic, the group of those pertaining to contrary-to-duty reasoning [16, 3, 5], and the group of those dealing with (unresolved) conflicts between obligations [4]. These two categories of problems are usually considered separately one from the other. We believe it is a virtue of the present framework that it covers them both.

We turn to the proof-theory. A derivation of a pair (a, x) from N, given a set R of rules, is understood to be a tree with (a, x) at the root, each non-leaf node related to its immediate parents by the inverse of a rule in R, and each leaf node an element of N.

Definition 2 (Proof system)**.** *$(a,x) \in D_1(N)$ if and only if (a,x) is derivable from N using the rules {SI, EQ, R-AND}*

$$SI\,\frac{(a,x) \qquad b \vdash a}{(b,x)}$$

$$EQ\,\frac{(a,x) \qquad x \dashv\vdash y}{(a,y)}$$

$$R\text{-}AND\,\frac{(a,x) \qquad (a,y)}{(a,x \wedge y)}\; a \wedge x \wedge y \textit{ is consistent}$$

Furthermore it is required that all the leaves of the derivation of (a,x) have (in the terminology of Makinson and van der Torre [8]) a consistent "fulfilment". That is, for all the leaves (b,y), $b \wedge y$ must be consistent.

When A is a set of formulas, $(A,x) \in D_1(N)$ means that $(a,x) \in D_1(N)$, for some conjunction a of elements of A. Furthermore, $D_1(N,A) = \{x : (A,x) \in D_1(N)\}$.

The requirement that the leaves of a derivation have a consistent fulfilment implies that D_1 fails inclusion, that is $N \subseteq D_1(N,A)$ does not necessarily hold. Put $N = \{(x,\neg x)\}$; we have $(x,\neg x) \notin D_1(N)$. This is in line with the semantics, which yields $(x,\neg x) \notin O_1(N)$.

The following applies.

Theorem 1 (Soundness and completeness)**.** $D_1(N,A) = O_1(N,A)$.

Proof. The proof is a re-run of the one given in Parent and van der Torre [14], suitably adapted to take into account the changes made to the definition of the I/O operation. □

3 Recycling the output as input

The system described in the previous section has an important limitation: it does not allow the output to be recycled as input. In other words, it cannot handle iteration of successive detachments. In this section, we show how to remove this limitation. We describe a semantics and a proof system, and we establish the soundness and completeness of the second with respect to the first.

We start with the semantics. The I/O operation is written O_3. We use the same subscript as Makinson and van der Torre [7]–our O_3 echoes their reusable I/O operation out_3.

Definition 3 (Semantics, iterated detachments)**.** *$x \in O_3(N, A)$ iff there is a finite $M \subseteq N$ and a set $B \subseteq Cn(A)$ such that $M(B) \neq \emptyset$, $x \dashv\vdash \wedge h(M)$, and*

i) $\forall B'(B \subseteq B' = Cn(B') \supseteq M(B') \Rightarrow b(M) \subseteq B')$

ii) $\{x\} \cup B$ *is consistent*

Observation 1. *Let M, B and B' be such that $B \subseteq B' \supseteq M(B') \cup b(M)$. We have $h(M) = M(B')$.*

Proof. The inclusion $M(B') \subseteq h(M)$ holds by definition. For the converse inclusion, let $y \in h(M)$. We have $(a, y) \in M$ for some $a \in b(M)$. Since $b(M) \subseteq B'$, $a \in B'$, and thus $y \in M(B')$ as required. □

As before, $O_3(N) = \{(A, x) : x \in O_3(N, A)\}$.

Observation 2. *If $x \in O_3(N, A)$ and $A \cup \{x\}$ is consistent, then there is a finite $M \subseteq N$ such that $M(Cn(A)) \neq \emptyset$, $x \dashv\vdash \wedge h(M)$ and*

$$\forall B'(Cn(A) \subseteq B' = Cn(B') \supseteq M(B') \Rightarrow b(M) \subseteq B') \tag{1}$$

Proof. Let $x \in O_3(N, A)$ and $A \cup \{x\}$ be consistent. By definition 3, there is a finite $M \subseteq N$ and a set $B \subseteq Cn(A)$ such that $M(B) \neq \emptyset$, $x \dashv\vdash \wedge h(M)$, and

i) $\forall B'(B \subseteq B' = Cn(B') \supseteq M(B') \Rightarrow b(M) \subseteq B')$

We have $M(Cn(A)) \neq \emptyset$, since $M(B) \neq \emptyset$. (1) follows from i) and $B \subseteq Cn(A)$. □

We now present the proof theory.

Definition 4 (Proof system)**.** *$(a, x) \in D_3(N)$ if and only if (a, x) is derivable from N using the rules {SI, EQ, R-ACT}:*

$$\textit{R-ACT}\ \frac{(a, x) \qquad (a \wedge x, y) \qquad a \wedge x \wedge y \text{ is consistent}}{(a, x \wedge y)}$$

As before it is required that all the leaves of the derivation of (a, x) have a consistent "fulfilment". That is, for all the leaves (b, y), $b \wedge y$ must be consistent.

As before, when A is a set of formulas, $(A, x) \in D_3(N)$ means that $(a, x) \in D_3(N)$ for some conjunction a of elements of A. Furthermore, $D_3(N, A) = \{x : (A, x) \in D_3(N)\}$.

The requirement that the leaves of a derivation have a consistent fulfilment implies that D_3 fails inclusion, that is $N \subseteq D_3(N)$ does not necessarily hold. Put $N = \{(x, \neg x)\}$; we have $(x, \neg x) \notin D_3(N)$. This is in line with the semantics, which yields $(x, \neg x) \notin O_3(N)$.

Observation 3. *O_3 (for an input formula a) verifies the rules of D_3.*

Proof. SI and EQ are straightforward. We show R-ACT. Assume

$$x \in O_3(N, a) \tag{HYP 1}$$

$$y \in O_3(N, a \wedge x) \tag{HYP 2}$$

$$a \wedge x \wedge y \text{ consistent} \tag{HYP 3}$$

From HYP 3, $a \wedge x$ is consistent. By observation 2, HYP 1 implies:

$$\begin{gathered}\exists M_1 \subseteq N \text{ such that } M_1(Cn(a)) \neq \emptyset \text{ and } x \dashv\vdash \wedge h(M_1) \text{ and} \\ \forall B'(Cn(a) \subseteq B' = Cn(B') \supseteq M_1(B') \Rightarrow b(M_1) \subseteq B')\end{gathered} \tag{2}$$

Similarly, by observation 2, HYP 2 and HYP 3 imply:

$$\begin{gathered}\exists M_2 \subseteq N \text{ such that } M_2(Cn(a, x)) \neq \emptyset \text{ and } y \dashv\vdash \wedge h(M_2) \text{ and} \\ \forall B'(Cn(a, x) \subseteq B' = Cn(B') \supseteq M_2(B') \Rightarrow b(M_2) \subseteq B')\end{gathered} \tag{3}$$

Put $M_3 = M_1 \cup M_2$. We have $M_3(Cn(a)) \neq \emptyset$. Also, $x \wedge y \dashv\vdash \wedge h(M_3)$.

Let B' be such that $Cn(a) \subseteq B' = Cn(B') \supseteq M_3(B')$. We have $B' \supseteq M_1(B')$. By (2), $b(M_1) \subseteq B'$. By observation 1, $x \dashv\vdash \wedge M_1(B')$. But $B' = Cn(B') \supseteq M_1(B')$. So $x \in B'$. Also $a \in B'$. Hence $a \wedge x \in B'$. So $Cn(a, x) \subseteq B'$. On the other hand, $B' \supseteq M_2(B')$. By (3), $b(M_2) \subseteq B'$, so that $b(M_3) \subseteq B'$. Last, $\{x \wedge y\} \cup Cn(a)$ is consistent, by HYP 3. Hence, $x \wedge y \in O_3(N, a)$ as required. □

Theorem 2 (Soundness)**.** *$D_3(N, A) \subseteq O_3(N, A)$.*

Proof. The proof follows the usual format in I/O logic, using theorem 3. The requirement that all the leaves of the derivation of (a, x) must have consistent fulfilment is needed to handle the case where (a, x) is in N. Details are omitted. □

The remainder of the paper is devoted to the proof of completeness.

Lemma 1. *If $x \in D_3(M, A)$, then $h(M) \vdash x$.*

Proof. By induction on the length of the derivation of (A, x). Details are omitted. □

Theorem 3 (Completeness). $O_3(N, A) \subseteq D_3(N, A)$.

Proof. Assume $x \in O_3(N, A)$, viz. $(A, x) \in O_3(N)$. There is a finite $M \subseteq N$ and a set $B \subseteq Cn(A)$ such that $M(B) \neq \emptyset$, $x \dashv\vdash \wedge h(M)$ and

i) $\forall B'(B \subseteq B' = Cn(B') \supseteq M(B') \Rightarrow b(M) \subseteq B')$

ii) $B \cup \{x\}$ is consistent

Define $B^\dagger = Cn(B \cup D_3(M, B))$.

Lemma 2. $M(B^\dagger) \subseteq B^\dagger$.

Proof of lemma 2. Let $y \in M(B^\dagger)$. Hence $(c, y) \in M$ and $c \in B^\dagger$. So $B \cup D_3(M, B) \vdash c$. Thus $b, y_1, ..., y_n \vdash c$, where b is a conjunction of elements in B, and $y_1, ..., y_n \in D_3(M, B)$. For all $i \leq n$, $y_i \in D_3(M, b_i)$, where b_i is a conjunction of elements in B. For the sake of conciseness, we define $\flat$ as a shorthand of $b \wedge (\wedge_{i=1}^n b_i)$. By PL, $\wedge_{i=1}^n y_i \vdash \flat \rightarrow c$, and thus $\wedge_{i=1}^n y_i \dashv\vdash \wedge_{i=1}^n y_i \wedge (\flat \rightarrow c)$.

Now, for two sub-lemmas.

Lemma 2.1. $\flat \wedge (\wedge_{i=1}^n y_i) \wedge (\flat \rightarrow c) \wedge y$ *is consistent.*

Proof of lemma 2.1. Proof by contradiction:

$\flat \wedge (\wedge_{i=1}^n y_i) \wedge (\flat \rightarrow c) \wedge y \vdash \bot$	assumption
$\flat \wedge (\wedge_{i=1}^n y_i) \wedge y \vdash \bot$	since $\wedge_{i=1}^n y_i \vdash \flat \rightarrow c$
$B \cup \{y_1, ..., y_n, y\} \vdash \bot$	since $B \vdash \flat$
$B \cup h(M) \cup \{y\} \vdash \bot$	by lemma 1
$B \cup h(M) \vdash \bot$	since $y \in h(M)$
$B \cup \{x\} \vdash \bot$	since $h(M) \dashv\vdash x$
= contradiction	

□

Lemma 2.2. $c \wedge y$ *is consistent.*

Proof of lemma 2.2. Proof by contradiction:

$c \wedge y \vdash \bot$	assumption
$\flat \wedge (\flat \to c) \wedge y \vdash \bot$	since $\flat \wedge (\flat \to c) \vdash c$
$=$ contradiction	

□

The argument for lemma 2 continues thus. Now, we have

$$\text{EQ}\ \dfrac{\dfrac{\dfrac{(b_1, y_1)}{(\flat, y_1)}\,\text{SI} \qquad \cdots \quad \cdots \qquad \dfrac{(b_n, y_n)}{(\flat, y_n)}\,\text{SI}}{(\flat, \wedge_{i=1}^{n} y_i)}\,\text{R-AND, lemma 2.1}}{(\flat, \wedge_{i=1}^{n} y_i \wedge (\flat \to c))}$$

Each (b_i, y_i) is the root of a derivation from leaves which (by definition) satisfy the requirement that they have a consistent fulfilment. Furthermore, the pair (c, y) has a consistent fulfilment, lemma 2.2. Thus,

$$\dfrac{(\flat, \wedge_{i=1}^{n} y_i \wedge (\flat \to c)) \qquad \dfrac{(c, y)}{(\flat \wedge (\wedge_{i=1}^{n} y_i) \wedge (\flat \to c), y)}\,\text{SI}}{(\flat, \wedge_{i=1}^{n} y_i \wedge (\flat \to c) \wedge y)}\,\text{R-ACT, lemma 2.1}$$

$\flat$ is a conjunction of formulas in B. This implies that

$$\wedge_{i=1}^{n} y_i \wedge (\flat \to c) \wedge y \in D_3(M, B)$$

and so $y \in B^{\dagger}$ as required.

This completes the proof of lemma 2. □

Lemma 3. $b(M) \subseteq B^{\dagger}$.

Proof of lemma 3. This follows from the fact that $B^{\dagger}$ meets all the conditions mentioned in the antecedent of the implication i). □

Lemma 4. $B \cup D_3(M, B)$*, and hence also* $B^{\dagger}$*, is consistent.*

Proof of lemma 4. We establish the claim for $B \cup D_3(M, B)$ by contradiction. In the following derivation, b_1, ... b_n are elements of $D_3(M, B)$.

$B \cup \{b_1, ..., b_n\} \vdash \bot$	assumption
$B \cup h(M) \vdash \bot$	by lemma 1
$B \cup \{x\} \vdash \bot$	since $h(M) \dashv\vdash x$
$=$ contradiction	

The claim for $B^\dagger$ follows from that for $B \cup D_3(M, B)$. □

Lemma 5. $h(M) \subseteq B^\dagger$.

Proof of lemma 5. This follows from observation 1, $h(M) = M(B^\dagger)$, combined with the above. □

Lemma 6. $b(M) \cup h(M)$ *is consistent.*

Proof of lemma 6. By lemmas 3 and 5, $b(M) \cup h(M) \subseteq B^\dagger$. By lemma 4, $B^\dagger$ is consistent. It immediately follows that $b(M) \cup h(M)$ is consistent. □

Lemma 7. $b(M) \cup \{x\}$ *is consistent.*

Proof of lemma 7. Immediate from lemma 6 and $x \dashv\vdash \wedge h(M)$. □

With lemmas 3 and 7 in hand, one then gets:

$x \in O_1(N, B \cup D_3(M, B))$	by definition 1
$x \in D_1(N, B \cup D_3(M, B))$	by theorem 1
$x \in D_3(N, B \cup D_3(M, B))$	

This means that $x \in D_3(N, \{b\} \cup \{a_1, ..., a_n\})$, where b is a conjunction of elements of B and, for each a_i, $a_i \in D_3(M, B)$. For each a_i, there is a conjunction b_i of elements in B such that $a_i \in D_3(M, b_i)$.

At this point, one last lemma is needed:

Lemma 8. $\wedge_{i=1}^n b_i \wedge (\wedge_{i=1}^n a_i) \wedge b \wedge x$ *is consistent.*

Proof of lemma 8. Proof by contradiction:

$\{a_1, ..., a_n, b_1, ..., b_n, b, x\} \vdash \bot$	assumption
$h(M) \cup \{b_1, ..., b_n, b, x\} \vdash \bot$	by lemma 1
$h(M) \cup B \cup \{x\} \vdash \bot$	since $b_1, ..., b_n, b \in Cn(B)$
$B \cup \{x\} \vdash \bot$	since $h(M) \dashv\vdash x$
= contradiction	

□

The following is thus derivable from M:

$$\dfrac{\dfrac{\dfrac{(b_1, a_1)}{(\wedge_{i=1}^n b_i, a_1)}\text{SI} \quad \cdots \quad \cdots \quad \dfrac{(b_n, a_n)}{(\wedge_{i=1}^n b_i, a_n)}\text{SI}}{(\wedge_{i=1}^n b_i, \wedge_{i=1}^n a_i)}\text{R-AND, lemma 8}}{(\wedge_{i=1}^n b_i \wedge b, \wedge_{i=1}^n a_i)}\text{SI}$$

The following is also derivable from N:

$$\dfrac{(b \wedge (\wedge_{i=1}^n a_i), x)}{(\wedge_{i=1}^n b_i \wedge b \wedge (\wedge_{i=1}^n a_i), x)}\text{SI}$$

By Lemma 1, for each a_i, $h(M) \vdash a_i$, and thus $h(M) \vdash \wedge_{i=1}^n a_i$. Hence, $x \vdash \wedge_{i=1}^n a_i$, and so $x \dashv\vdash x \wedge (\wedge_{i=1}^n a_i)$. Furthermore, $a^\star \vdash \wedge_{i=1}^n b_i \wedge b$, where $a^\star$ is a conjunction of elements of A. The following may, then, be derived.

$$\text{SI}\,\dfrac{\text{EQ}\,\dfrac{\text{R-ACT, lemma 8}\,\dfrac{(\wedge_{i=1}^n b_i \wedge b, \wedge_{i=1}^n a_i) \qquad (\wedge_{i=1}^n b_i \wedge b \wedge (\wedge_{i=1}^n a_i), x)}{(\wedge_{i=1}^n b_i \wedge b, \wedge_{i=1}^n a_i \wedge x)}}{(\wedge_{i=1}^n b_i \wedge b, x)}}{(a^\star, x)}$$

Since $a^\star$ is a conjunction of elements of A, the pair (A, x) is derivable. □

This completes the proof of the main result of this paper, theorem 3.

4 Topics for future research

We end this paper with a number of topics for future research.

- Other restricted forms of chaining can be considered, like

$$\text{R-AT}\,\dfrac{(a, x) \qquad (x, y) \qquad a \wedge x \text{ (resp. } x \wedge y\text{) is consistent}}{(a, x \wedge y)}$$

  R-AT is short for "Restricted aggregative transitivity". Is there an I/O operation validating this rule?

- In constrained I/O logic, there is the idea of a constraint set C filtering excess output. In the present paper, only the body and the head of a rule is treated as a constraint. Could one generalize the I/O logics with a consistency check in such a way that one can also work with an independent "constraint set"? Parent [10] and Dustin [20] use this technique to model defeasible reasoning. They take the traditional I/O logics as a starting point. What happens if the system described in the present paper is taken as a starting point? Would it yield new insights into our understanting of defeasible reasoning?

- What about the I/O logics for positive (static and dynamic) permission described by Makinson and van der Torre [9]? The various meta-results they establish in their paper (like the axiomatisation of positive static permission with the subverse rules) hold if a traditional I/O logic for obligation is used. Do similar meta-results can be obtained, using the system described in the present paper?

- Benzmüller and Parent [1] report some first results regarding the question of how to "implement" I/O logic using the so-called Shallow Semantic Embedding approach developed by Benzmüller. Their focus is on the traditional I/O logics. Can a similar embedding be obtained for the system described in the present paper?

References

[1] C. Benzmüller and X. Parent. I/O logic in HOL — first steps. Technical report, CoRR, 2018. `https://arxiv.org/abs/1803.09681`.

[2] J. Broersen and L. van der Torre. Ten problems of deontic logic and normative reasoning in computer science. In N. Bezhanishvili and V. Goranko, editors, *Lectures on Logic and Computation*, volume 7388 of *Lecture Notes in Computer Science*, pages 55–88. Springer, Berlin, Heidelberg, 2012.

[3] J. Carmo and A. J. I. Jones. Deontic logic and contrary-to-duties. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic: Volume 8*, pages 265–343. Springer Netherlands, Dordrecht, 2002.

[4] L. Goble. Prima facie norms, normative conflicts, and dilemmas. In D. Gabbay, J. Horty, X. Parent, R. van der Meyden, and L. van der Torre, editors, *Handbook of Deontic Logic and Normative Systems*, pages 241–352. College Publications, London. UK, 2013.

[5] G. Governatori and A. Rotolo. Logic of violations: A Gentzen system for reasoning with contrary-to-duty obligations. *Australasian Journal of Logic*, 3, 2005.

[6] J. Hansen. Reasoning about permission and obligation. In S. O. Hansson, editor, *David Makinson on Classical Methods for Non-Classical Problems*, pages 287–333. Springer, 2014.

[7] D. Makinson and L. van der Torre. Input/output logics. *Journal of Philosophical Logic*, 29(4):383–408, 2000.

[8] D. Makinson and L. van der Torre. Constraints for input/output logics. *Journal of Philosophical Logic*, 30(2):155–185, 2001.

[9] D. Makinson and L. van der Torre. Permission from an input/output perspective. *Journal of Philosophical Logic*, 32(4):391–416, 2003.

[10] X. Parent. Moral particularism in the light of deontic logic. *Artif. Intell. Law*, 19(2-3):75–98, 2011.

[11] X. Parent, D. Gabbay, and L. van der Torre. Intuitionistic basis for input/output logic. In Sven Ove Hansson, editor, *David Makinson on Classical Methods for Non-Classical Problems*, pages 263–286. Springer Netherlands, Dordrecht, 2014.

[12] X. Parent and L. van der Torre. "Sing and dance!". In F. Cariani, D. Grossi, J. Meheus, and X. Parent, editors, *Deontic Logic and Normative Systems*, pages 149–165, Cham, 2014. Springer International Publishing.

[13] X. Parent and L. van der Torre. Detachment in normative systems: examples, inference patterns, properties. In G. Pigozzi and L. van der Torre (guest editors), editors, *IfCoLog Journal of Logics and their Applications, Volume 4 (9)*, volume 4, pages 2295–3038. 2017.

[14] X. Parent and L. van der Torre. The pragmatic oddity in a norm-based semantics. In G. Governatori, editor, *Proceedings of the 16th Edition of the International Conference on Artificial Intelligence and Law*, ICAIL '17, pages 169–178, New York, NY, USA, 2017. ACM.

[15] H. Prakken and M. Sergot. Contrary-to-duty obligations. *Studia Logica*, 57(1):91–115, 1996.

[16] H. Prakken and M. Sergot. Dyadic deontic logic and contrary-to-duty obligations. In D. Nute, editor, *Defeasible Deontic Logic*, pages 223–262. Springer Netherlands, Dordrecht, 1997.

[17] A. Stolpe. Normative consequence: The problem of keeping it whilst giving it up. In R. van der Meyden and L. van der Torre, editors, *Deontic Logic in Computer Science, 9th International Conference, DEON 2008*, volume 5076 of *Lecture Notes in Computer Science*, pages 174–188. Springer, 2008.

[18] A. Stolpe. *Norms and Norm-System Dynamics*. PhD thesis, Department of Philosophy, University of Bergen, Norway, 2008.

[19] A. Stolpe. A concept approach to input/output logic. *J. of Applied Logic*, 13(3):239–258, 2015.

[20] D. Tucker. Variable priorities and exclusionary reasons in input/output logic. To appear in *Journal of Philosophical Logic*.

Received \jreceived