Intuitionistic Basis for Input/Output Logic

Xavier Parent, Dov Gabbay and Leendert van der Torre

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Abstract It is shown that I/O logic can be based on intuitionistic logic instead of classical logic. More specifically, it is established that, when going intuitionistic, a representation theorem is still available for three of the four (unconstrained) original I/O operations. The trick is to see a maximal consistent set as a saturated one. The axiomatic characterization is as in the classical case. Therefore, the choice between the two logics does not make any difference for the resulting framework.

Key words: deontic logic; I/O Logic; intuitionistic logic; completeness; saturated set

1 Introduction

So-called input/output logic (I/O logic, for short) is a general framework devised by Makinson and van der Torre [9, 10, 11] in order to reason about conditional norms. A frequent belief about I/O logic is that it presupposes classical logic. The aim of this paper is to show that this is a misunderstanding.

From the outset Makinson and van der Torre made it clear that it would be quite misleading to refer to I/O logic as a kind of 'non-classical logic.' Its role, they argue,

Xavier Parent
University of Luxembourg, e-mail: xavier.parent@uni.lu

Dov Gabbay
Bar-Ilan University Israel, King's College London and University of Luxembourg, e-mail: dov.gabbay@kcl.ac.uk

Leendert van der Torre
University of Luxembourg, e-mail: leon.vandertorre@uni.lu

is to study not some kind of non-classical logic, but "a way of using the classical one" [9, p. 384]. The basic intuition motivating the I/O framework is explained further thus, by contrasting an old and a new way to think about logic:

> "From a very general perspective, logic is often seen as an 'inference motor', with premises as inputs and conclusions as outputs [...]. But it may also be seen in another role, as 'secretarial assistant' to some other, perhaps non-logical, transformation engine [...]. From this point of view, the task of logic is one of preparing inputs before they go into the machine, unpacking outputs as they emerge and, less obviously, co-ordinating the two." [9, p. 384]

In input/output logic, the meaning of the deontic concepts is given in terms of a set of procedures yielding outputs for inputs. Thus, the semantics may be called "operational" rather than truth-functional. This is where the black box analogy comes in. To some extent, the system can be viewed solely in terms of its input, output and transfer characteristics without any knowledge of its internal workings, which remain "opaque" (black). Logic is here reduced to an ancillary role in relation to it.

The picture of logic assisting a transformation engine appears to be very general: *prima facie* any base logic may act as a secretarial assistant. Still, the only input-output logics investigated so far in the literature are built on top of classical propositional logic. In particular, one of the key building blocks is the notion of maximal consistent set. In the context of logics other than classical logic, this notion will not do the required job. This raises the question of whether the framework is as general as one might think at first sight, and whether it can be instantiated using other logics. If the answer is positive, then one would like to know what properties of classical logic are essential for the completeness result to obtain.

As a first step towards clarifying these issues, we here consider the case of intuitionistic logic. The main observation of this paper is that, if I/O logic is built on top of the latter, then a representation theorem for three of the four standard I/O operations is still available. These are the so-called simple-minded, basic, and simple-minded reusable operations. What is more, the axiomatic characterization of the I/O operations remains the same. We will show that the job done by the notion of maximal consistent set in I/O logic can equally be done by its analog within intuitionistic logic, the notion of saturated set. The result is given for the unconstrained version of I/O logics dealing with obligation. It is the only one that comes with a known complete axiomatic characterization. The unconstrained version of I/O logics dealing with permission (see [11]), and the constrained version of I/O logics dealing with contrary-to-duties (see [10]) both come with a syntactical characterization or proof-theory. But strictly speaking this one is not an axiomatization. There are no soundness and completeness theorems relating operations to proof systems.

We also complement our positive results by a negative one concerning the fourth and last standard operation, called basic reusable. It is pointed out that the so-called rule of "ghost contraposition", which holds for the classically based operation, fails for its intuitionistic counterpart. The axiomatization problem for the latter I/O operation remains an open one.

This paper is organized as follows. Section 2 provides the required background. Section 3 gives the completeness results. Section 4 concludes, and provides some suggestions as to useful ways forward.

2 Background

Subsection 2.1 presents the framework as initially defined by Makinson and van der Torre [9]. We shall refer to it as the initial framework. Subsection 2.2 explains how to change the base logic from classical to intuitionistic logic.

2.1 I/O Logic

In input/output logic, a normative code is a (non-empty) set G of conditional norms, which is a (non-empty) set of ordered pairs (a, x). Here a and x are two well-formed formulae (wff's) of propositional logic. Each such pair may be referred to as a generator. The body a is thought of as an input, representing some condition or situation, and the head x is thought of as an output, representing what the norm tells us to be obligatory in that situation.

Some further notation. L is the set of all formulae of propositional logic. Given an input $A \subset L$, and a set of generators G, $G(A)$ denotes the image of G under A, i.e., $G(A) = \{x : (a,x) \in G \text{ for some } a \in A\}$. $Cn(A)$ denotes the set $\{x : A \vdash x\}$, where $\vdash$ is the consequence relation used in classical logic. The notation $x \dashv\vdash y$ is short for $x \vdash y$ and $y \vdash x$.

2.1.1 Semantics

As mentioned, I/O logic comes with an operational rather than truth-functional semantics. The meaning of the deontic concepts is given in terms of a set of procedures yielding outputs for inputs. The following I/O operations can be defined.

Definition 1 (I/O operations). Let A be an input set, and let G be a set of generators. The following input/output operations can be defined, where a complete set is one that is either maximal consistent[1] or equal to L:

$$out_1(G,A) = Cn(G(Cn(A)))$$
$$out_2(G,A) = \cap\{Cn(G(V)) : A \subseteq V, V complete\}$$
$$out_3(G,A) = \cap\{Cn(G(B)) : A \subseteq B \supseteq Cn(B) \supseteq G(B)\}$$
$$out_4(G,A) = \cap\{Cn(G(V)) : A \subseteq V \supseteq G(V), V complete\}$$

$out_1(G,A), out_2(G,A), out_3(G,A)$ and $out_4(G,A)$ are called "simple-minded" output, "basic" output, "simple-minded reusable" output, and "basic reusable" output, respectively.

[1] The set is consistent, and none of its proper extensions is consistent.

These four operations have four counterparts that also allow throughput. Intuitively, this amounts to requiring $A \subseteq out_i(G,A)$ for $i \in \{1,2,3,4\}$. In terms of the definitions, it is to require that G is expanded to contain the diagonal, i.e., all pairs (a,a). The throughput version of the I/O operations will be put to one side in this preliminary study.

2.1.2 Proof-theory

Put $out_i(G) = \{(A,x) : x \in out_i(G,A)\}$ for $i \in \{1,2,3,4\}$. This definition leads to an axiomatic characterization that is much alike those used for conditional logic.

The specific rules of interest here are described below. They are formulated for a singleton input set A (for such an input set, curly brackets will be omitted). The move to an input set A of arbitrary cardinality will be explained in a moment.

$$\text{(SI)} \quad \frac{(a,x) \qquad b \vdash a}{(b,x)}$$

$$\text{(WO)} \quad \frac{(a,x) \qquad x \vdash y}{(a,y)}$$

$$\text{(AND)} \quad \frac{(a,x) \qquad (a,y)}{(a,x \wedge y)}$$

$$\text{(OR)} \quad \frac{(a,x) \qquad (b,x)}{(a \vee b,x)}$$

$$\text{(CT)} \quad \frac{(a,x) \qquad (a \wedge x,y)}{(a,y)}$$

SI, WO, and CT abbreviate “strengthening of the input”, “weakening of the output”, and “cumulative transitivity” respectively.

Given a set R of rules, a derivation from a set G of pairs (a,x) is a sequence α_1,.., α_n of pairs of formulae such that for each index $0 \leq i \leq n$ one of the following holds:

- α_i is an hypothesis, i.e. $\alpha_i \in G$;
- α_i is $(\top,\top)$, where $\top$ is a zero-place connective standing for ‘tautology’;
- α_i is obtained from preceding element(s) in the sequence using a rule from R.

The elements in the sequence are all pairs of the form (a,x). Derivation steps done in the base logic are not part of it.

A pair (a,x) of formulae is said to be derivable from G if there is a derivation from G whose final term is (a,x). This may be written as $(a,x) \in deriv(G)$, or equivalently $x \in deriv(G,a)$. A subscript will be used to indicate the set of rules employed. The specific systems of interest here will be referred to as $deriv_1$, $deriv_2$, $deriv_3$, and $deriv_4$. They are defined by the rules {SI, WO, AND}, {SI, WO, AND, OR}, {SI, WO, AND, CT}, and {SI, WO, AND, OR, CT}, respectively. Note that, given (SI), (CT) can be strengthened into plain transitivity (“From (a,x) and (x,y), infer (a,y)”).

When A is a set of formulae, derivability of (A,x) from G is defined as derivability of (a,x) from G for some conjunction $a = a_1 \wedge ... \wedge a_n$ of elements of A. We understand the conjunction of zero formulae to be a tautology, so that $(\emptyset,a)$ is derivable from G iff $(\top,a)$ is.

The following applies:

Theorem 1 (Soundness and completeness).

$$out_1(G,A) = deriv_1(G,A)$$
$$out_2(G,A) = deriv_2(G,A)$$
$$out_3(G,A) = deriv_3(G,A)$$
$$out_4(G,A) = deriv_4(G,A)$$

Proof. See Makinson and van der Torre [9]. □

Table 1 shows the output operations, and the rules corresponding to them.

Output operation	Rules
Simple-minded (out_1)	{SI, WO, AND}
Basic (out_2)	{SI, WO, AND}+{OR}
Simple-minded reusable (out_3)	{SI, WO, AND}+{CT}
Basic reusable (out_4)	{SI, WO, AND}+{OR,CT}

Table 1 I/O systems

2.2 *Intuitionistic I/O logic*

This subsection describes the changes that must be made to I/O logic for it to be based on intuitionistic logic. We start with a few highlights on the latter. Those readers already familiar with intuitionistic logic and its Kripke semantics should pass directly to §2.2.2, returning to §2.2.1 if needed for specific points.

2.2.1 Intuitionistic logic

Intuitionistic logic has its roots in the philosophy of mathematics propounded by Brouwer in the early twentieth century. According to Brouwer, truth in mathematics means the existence of a proof (cf [15].) Intuitionistic logic can, thus, be described as departing from classical logic in its definition of the meaning of a statement being true. In classical logic, a statement is ‘true’ or ‘false’, independently of our knowledge concerning the statement in question. In intuitionistic logic, a statement is 'true' if it has been proved, and 'false' if it has been disproved (in the sense that there is a proof that the statement is false). As a result, intuitionistic logicians reject two laws of classical logic, among others. One is the law of excluded middle, $a \vee \neg a$, and the other is the law of double negation elimination, $\neg\neg a \to a$. Under the intuitionistic reading, an assertion of the form $a \vee \neg a$ implies the ability to either prove or refute a. And a statement like $\neg\neg a$ says that a refutation of a has been disproved. You may have an opinion that this is not the same as a proof (or reinstatement) of a; the truth of a may still be uncertain. Of course, even though intuitionistic logic was initially conceived as the correct logic to apply in mathematical reasoning, it would be a mistake to restrict the latter logic to the mathematical domain. The intuitionistic understanding of mathematical language may be generalized to all language, if one takes the notion of proof in a very broad sense. Most notably, Dummett (see e.g. [1]) championed such a generalization. The key idea is to assume that the meaning of a statement is always given by its justification conditions, i.e. the conditions under which one would be justified in accepting the statement. This is known as the verificationist theory of meaning.

In this paper, intuitionistic logic will be described as the minimal logic that both contains the *ex falso* rule

(Ex falso) $$a, \neg a \vdash b$$

and allows for the deduction theorem:

(DT) $$\Gamma \vdash a \to b \text{ iff } \Gamma, a \vdash b$$

Classical logic allows for a stronger equivalence, namely:

(SDT) $$\Gamma \vdash (a \to b) \vee c \text{ iff } \Gamma, a \vdash b \vee c$$

Below we list the main properties of $\vdash$ to which we shall appeal later. They are taken from the discussion by Thomason [14]. From now onwards, $\vdash$ and *Cn* will refer to intuitionistic rather than classical logic.

Group I

(**Ref**) If $a \in \Gamma$, then $\Gamma \vdash a$

(**Mon**) $$\frac{\Gamma \vdash a}{\Gamma \cup \Delta \vdash a}$$

(**Cut**) $$\frac{\Gamma \vdash a \qquad \Gamma \cup \{a\} \vdash b}{\Gamma \vdash b}$$

(**C**) If $\Gamma \vdash a$, then $\Gamma' \vdash a$ for some finite $\Gamma' \subseteq \Gamma$

The labels (**Ref**), (**Mon**), and (**C**) are mnemonic for “Reflexivity”, “Monotony”, and “Compactness”, respectively. (**Cut**) can be equivalently expressed as

$$(\textbf{Cut'})\ \frac{\Gamma \vdash a \text{ for all } a \in A \qquad \Gamma \cup A \vdash b}{\Gamma \vdash b}$$

Group II

$$(\to\!:\!\textbf{I})\ \frac{\Gamma \cup \{a\} \vdash b}{\Gamma \vdash a \to b} \qquad (\to\!:\!\textbf{E})\ \frac{\Gamma \vdash a \qquad \Gamma \vdash a \to b}{\Gamma \vdash b}$$

$$(\vee\!:\!\textbf{I})\ \frac{\Gamma \vdash a}{\Gamma \vdash a \vee b}\ \frac{\Gamma \vdash b}{\Gamma \vdash a \vee b} \quad (\vee\!:\!\textbf{E})\ \frac{\Gamma \cup \{a\} \vdash c \qquad \Gamma \cup \{b\} \vdash c \qquad \Gamma \vdash a \vee b}{\Gamma \vdash c}$$

$$(\wedge\!:\!\textbf{I})\ \frac{\Gamma \vdash a \qquad \Gamma \vdash b}{\Gamma \vdash a \wedge b} \qquad (\wedge\!:\!\textbf{E})\ \frac{\Gamma \vdash a \wedge b}{\Gamma \vdash a}\ \frac{\Gamma \vdash a \wedge b}{\Gamma \vdash b}$$

$$(\neg\!:\!\textbf{I})\ \frac{\Gamma \cup \{a\} \vdash b \qquad \Gamma \cup \{a\} \vdash \neg b}{\Gamma \vdash \neg a} \qquad (\neg\!:\!\textbf{E})\ \frac{\Gamma \vdash a \qquad \Gamma \vdash \neg a}{\Gamma \vdash b}$$

$$(\top\!:\!\textbf{I})\ \frac{-}{\Gamma \vdash \top}$$

The rules in group I may be called “structural”. They are also often referred to as the “Tarski conditions” in honor of Alfred Tarski who first saw their importance. Indeed these can be shown to express conditions that are jointly necessary and sufficient for propositions to be chained together in a derivation (see Makinson [8, §10.2]).

The rules in group II may be called “elementary”. They can be classified as introduction or elimination rules depending on whether they allow us to introduce or eliminate a connective. $\top$ has no elimination rule.

The principle (Ex falso) follows from (**Ref**) and ($\neg$:**E**). The reader may also easily verify that (DT) is derivable. One direction is just ($\to$:**I**). For the other, assume $\Gamma \vdash a \to b$. By (**Mon**), $\Gamma, a \vdash a \to b$. By (**Ref**) and (**Mon**), $\Gamma, a \vdash a$. By ($\to$:**E**), $\Gamma, a \vdash b$, as required.

Negation in classical logic (so-called classical negation) is defined by ($\neg$:**I**) and the rule

$$(\neg\!:\!\textbf{E'})\ \frac{\Gamma, \neg a \vdash b \qquad \Gamma, \neg a \vdash \neg b}{\Gamma \vdash a}$$

In the presence of ($\neg$:**E'**), the laws $\vdash a \vee \neg a$ and $\vdash \neg\neg a \to a$ become derivable. It is noteworthy that (SDT) becomes derivable too. The proofs are omitted.

We write $x[y/a]$ for the formula obtained by replacing, in x, all occurrences of atom a with y. Where Γ is a set of formulae, we write $\Gamma[y/a]$ for $\{x[y/a] : x \in \Gamma\}$.

Theorem 2 (Substitution theorem for derivability). *If* $\Gamma \vdash x$, *then* $\Gamma[y/a] \vdash x[y/a]$.

Proof. This is [16, Theorem 5.2.4]. □

If some formula y occurs in a formula x, the occurrence itself can be described by a sequence of natural numbers. It gives the so-called position of y in (the construction tree of) x. For instance, the second occurrence of b has position 22 in $(a \wedge b) \rightarrow (a \vee b)$. We write $x[y]_p$ for the result of replacing the subformula at position p in x by y. The formal definition of the notion of position is as usual, and so is that of the '$x[y]_p$' construct.[2] We have:

Theorem 3 (Replacement of equivalents). *If* $x_1 \dashv\vdash x_2$*, then* $x[x_1]_p \dashv\vdash x[x_2]_p$.

Proof. The proof is standard, and is omitted. □

For future reference, we note the following facts.

Lemma 1.

(1) $a \vdash a \vee b$ *and* $b \vdash a \vee b$

(2) $a \wedge b \vdash a$ *and* $a \wedge b \vdash b$

(3) $a \vdash b \rightarrow a$

(4) *If* $a \vdash c$ *and* $b \vdash c$ *then* $a \vee b \vdash c$

(5) *If* $a \vdash c$ *then* $a \wedge b \vdash c$

(6) $a, b \vdash c$ *iff* $a \wedge b \vdash c$

(7) $a \dashv\vdash a \wedge \top$

(8) $\top \vdash x$ *whenever* $x \in Cn(\emptyset)$

(9) $a \wedge (a \rightarrow b) \vdash b$

(10) $a \wedge b \wedge (c \vee d) \vdash (a \wedge c) \vee (b \wedge d)$

(11) $Cn(\Gamma) = CnCn(\Gamma)$

Proof. For (1), we have $a \vdash a$ by (**Ref**). Using ($\vee$:**I**) it follows that $a \vdash a \vee b$. The argument for $b \vdash a \vee b$ is similar.

For (2), by (**Ref**), $a \wedge b \vdash a \wedge b$. By ($\wedge$:**E**), $a \wedge b \vdash a$. The argument for $a \wedge b \vdash b$ is the same.

For (3), by (**Ref**), $a \vdash a$. By (**Mon**), $a, b \vdash a$. By ($\rightarrow$:**I**), $a \vdash b \rightarrow a$.

For (4), assume $a \vdash c$ and $b \vdash c$. By (**Mon**), $a \vee b, a \vdash c$ and $a \vee b, b \vdash c$. By (**Ref**), $a \vee b \vdash a \vee b$. By ($\vee$:**E**), $a \vee b \vdash c$ as required.

For (5), assume $a \vdash c$. By (**Mon**), $a \wedge b, a \vdash c$. By (2) in this Lemma, $a \wedge b \vdash a$. By (**Cut**), $a \wedge b \vdash c$.

For (6). From left-to-right, assume $a, b \vdash c$. By (**Mon**), $a \wedge b, a, b \vdash c$. By (2) in this Lemma, $a \wedge b \vdash a$, and $a \wedge b \vdash b$. So, by (**Cut'**), $a \wedge b \vdash c$ as required. From right-to-left, assume $a \wedge b \vdash c$. By (**Ref**) and (**Mon**), $a, b \vdash a$ and $a, b \vdash b$. By ($\wedge$:**I**), $a, b \vdash a \wedge b$. By (**Mon**), $a, b, a \wedge b \vdash c$. By (**Cut**), $a, b \vdash c$.

For (7), $a \wedge \top \vdash a$ is just a special case of (2). By (**Ref**), $a \vdash a$. By ($\top$:**I**), $a \vdash \top$. By ($\wedge$:**I**), $a \vdash a \wedge \top$.

For (8), assume $x \in Cn(\emptyset)$. So $\emptyset \vdash x$. By (**Mon**), $\top \vdash x$ as required.

[2] See, for example, [12, §2].

For (9). By (2) in this Lemma, $a \wedge (a \to b) \vdash a$ and $a \wedge (a \to b) \vdash a \to b$. By ($\to$:**E**), $a \wedge (a \to b) \vdash b$ as required.

For (10), using (2) in this Lemma and (**Mon**), we get $c, a \wedge b \vdash a$. By (**Ref**) and (**Mon**), $c, a \wedge b \vdash c$. So $c, a \wedge b \vdash a \wedge c$ by ($\wedge$:**I**). And, by (1) in this Lemma, $a \wedge c \vdash (a \wedge c) \vee (b \wedge d)$. By (**Mon**), $c, a \wedge b, a \wedge c \vdash (a \wedge c) \vee (b \wedge d)$. By (**Cut**), $c, a \wedge b \vdash (a \wedge c) \vee (b \wedge d)$. By ($\to$:**I**), $c \vdash (a \wedge b) \to ((a \wedge c) \vee (b \wedge d))$. A similar argument yields $d \vdash (a \wedge b) \to ((a \wedge c) \vee (b \wedge d))$. By (4) in this Lemma, $c \vee d \vdash (a \wedge b) \to ((a \wedge c) \vee (b \wedge d))$. Using the left-to-right direction of (DT), we get $c \vee d, a \wedge b \vdash (a \wedge c) \vee (b \wedge d)$. After rearranging the premisses, the conclusion $(a \wedge b) \wedge (c \vee d) \vdash (a \wedge c) \vee (b \wedge d)$ follows, by a straightforward application of (6) in this Lemma.

(11) is the property of idempotence for Cn. The left-in-right inclusion follows from (**Ref**). For the right-in-left inclusion, let $x \in CnCn(\Gamma)$. By (**C**), there are $x_1, \ldots, x_n (n \geq 0)$ in $Cn(\Gamma)$ such that $x_1, \ldots, x_n \vdash x$. On the one hand, $\Gamma \vdash x_1$, ..., and $\Gamma \vdash x_n$. On the other hand, $\Gamma \cup \{x_1, \ldots, x_n\} \vdash x$ by (**Mon**). So $\Gamma \vdash x$ by (**Cut'**). Hence $x \in Cn(\Gamma)$, as required. □

It is a well-known fact that intuitionistic logic satisfies the so-called disjunction property. Expressed in terms of theoremhood, this is the property that, if $\vdash a \vee b$, then $\vdash a$ or $\vdash b$. This property also holds for the notion of deducibility under a set of assumptions, but in a qualified form. A restriction must be placed on the occurrences of $\vee$ in the premisses set Γ. The proviso in question may be formulated in terms of the notion of Harrop formula. The Harrop formulae, named after Ronald Harrop [5], are the class of formulae defined inductively as follows:

- $a \in H$ for atomic a;
- $\neg a \in H$ for any wff a;
- $a, b \in H \Rightarrow a \wedge b \in H$;
- $b \in H \Rightarrow a \to b \in H$.

Intuitively, a Harrop formula is a formula in which all the disjunctions are 'hidden' inside the left-hand scope of an implication, or inside the scope of a negation.

We say that a set Γ is Harrop if it is made up of Harrop formulae only.

Theorem 4 (Disjunction property under hypothesis). *If Γ is Harrop, then, if $\Gamma \vdash a \vee b$, it follows that $\Gamma \vdash a$ or $\Gamma \vdash b$.*

Proof. This is Theorem 23 in [17]. For a proof, see [2, Ch. 2, §3]. □

In intuitionistic logic, a set Γ of wff's is said to be consistent − written $\mathrm{Con}\,\Gamma$ − just in case $\Gamma \nvdash a$ for some wff a. Thus, Γ is inconsistent − $\mathrm{Incon}\,\Gamma$ − just when $\Gamma \vdash a$ for all wff a. For future reference, Theorem 5 records some properties of the notion of consistency thus conceived.

Theorem 5.

(1) $\mathrm{Con}\{a \vee b\}$ *iff:* $\mathrm{Con}\{a\}$ *or* $\mathrm{Con}\{b\}$

(2) $\mathrm{Con}\{b\}$ *implies* $\mathrm{Con}\{a\}$ *whenever* $b \vdash a$

(3) *$Con\,\Gamma$ iff $Con\,\Gamma'$ for all finite $\Gamma' \subseteq \Gamma$*

(4) *$Con\,\Gamma$ implies $Con\,Cn(\Gamma)$*

Proof. For (1):

- For the left-to-right direction, assume $\mathrm{Con}\{a \vee b\}$. This means that $\{a \vee b\} \nvdash c$ for some c. By Lemma 1 (4), either $\{a\} \nvdash c$ or $\{b\} \nvdash c$, which suffices for $\mathrm{Con}\{a\}$ or $\mathrm{Con}\{b\}$.
- For the right-to-left direction, we show the contrapositive. Let $\mathrm{Incon}\{a \vee b\}$. So $a \vee b \vdash c$ for all c. Consider an arbitrary c. By Lemma 1 (1), $a \vdash a \vee b$. By (**Mon**), $a, a \vee b \vdash c$. By (**Cut**), $a \vdash c$. So, $a \vdash c$ for all c, and thus $\mathrm{Incon}\{a\}$. The argument for $\mathrm{Incon}\{b\}$ is similar.

For (2), assume $b \vdash a$ and $\mathrm{Con}\{b\}$. The latter means that $b \nvdash c$ for some c. From this and $b \vdash a$ using (**Cut**) it follows that $a, b \nvdash c$. By (**Mon**), $a \nvdash c$, which suffices for $\mathrm{Con}\{a\}$.
For (3). This uses consistency to express compactness.

- For the left-to-right direction, assume $\mathrm{Con}\,\Gamma$. So, $\Gamma \nvdash c$ for some c. Let Γ' be a finite subset of Γ. By (**Mon**), $\Gamma' \nvdash c$, and thus $\mathrm{Con}\,\Gamma'$ as required.
- For the reverse, assume $\mathrm{Con}\,\Gamma'$ for any finite $\Gamma' \subseteq \Gamma$, but $\mathrm{Incon}\,\Gamma$. From the latter, $\Gamma \vdash c$ and $\Gamma \vdash \neg c$ for an arbitrarily chosen c. By (**C**), $\Gamma_1 \vdash c$ and $\Gamma_2 \vdash \neg c$ for some finite $\Gamma_1, \Gamma_2 \subseteq \Gamma$. Since Γ_1 and Γ_2 are both finite, so is $\Gamma_1 \cup \Gamma_2$. Furthermore, by (**Mon**), $\Gamma_1 \cup \Gamma_2 \vdash c$ and $\Gamma_1 \cup \Gamma_2 \vdash \neg c$. By ($\neg$:**E**), $\Gamma_1 \cup \Gamma_2 \vdash b$ for all b. So $\mathrm{Incon}\,(\Gamma_1 \cup \Gamma_2)$, contradicting the opening assumption.

For (4), assume $\mathrm{Con}\,\Gamma$. So $c \notin Cn(\Gamma)$ for some wff c. By Lemma 1 (11), $c \notin CnCn(\Gamma)$, and thus $\mathrm{Con}\,Cn(\Gamma)$ as required. □

We end this section by describing a Kripke-type semantics commonly used for intuitionistic logic. A Kripke model M for intuitionistic logic is a triplet $(W, \geq, V)$, where

- W is a set of possible worlds, t, s, ...;
- $\geq$ is a reflexive and transitive relation on W;
- V an evaluation function assigning to each propositional letter a the set of worlds at which a is true.

Each world t forces the truth of formulae, and this relation is indicated by $\models$. Following Kripke [7], one might think of the worlds as representing points in time (or "evidential situations"), at which we may have various pieces of information. If, at a given time point t, we have enough information to prove x, then we say that x has been verified at t, or that t forces x. If we lack such information, we say that x has not been verified at t, or that t does not force x. Persistence over time is required for all atomic a, in the sense that

$$t \geq s \text{ and } s \in V(a) \text{ imply } t \in V(a)$$

The forcing relation $\models$ satisfies the usual conditions except for

$$w \models x \to y \text{ iff } \forall w'((w' \geq w \,\&\, w' \models x) \Rightarrow w' \models y)$$
$$w \models \neg x \text{ iff } \neg\exists w'(w' \geq w \,\&\, w' \models x)$$

The notions of semantic consequence and satisfiability are defined in the usual way. For a formula x to be a semantic consequence of Γ (notation: $\Gamma \models x$), it must be the case that, whenever all the formulae in Γ are forced at some point in a model, then x is forced in that same model at the same point. A set Γ of formulae is said to be satisfiable if there is some model in which all its component formulae are forced at some point in the model. We have:

Theorem 6 (Soundness and completeness). $\Gamma \vdash x$ *iff* $\Gamma \models x$

Proof. See e.g. [14]. □

For more on intuitionistic logic, the reader is referred to Gabbay [2, 3].

2.2.2 Redefining the I/O operations

For out_1 and out_3, it is natural to keep the same definitions as in the original framework, but assume that the underlying consequence relation Cn is defined as in intuitionistic logic.

For out_2 and out_4, a little more care is needed. The original account uses the notion of maximal consistent set. If the base logic is intuitionistic logic, the latter notion will not be suitable for the problem at hand. We use the notion of saturated set instead. It can be defined thus.

Definition 2 (Saturated set, [14]). Let S be a (non-empty) set of wff's. S is said to be saturated if the following three conditions holds:

(1) $\mathrm{Con}\, S$ (S is consistent)

(2) $a \vee b \in S \Rightarrow a \in S$ or $b \in S$ (S is join-prime)

(3) $S \vdash a \Rightarrow a \in S$ (S is closed under $\vdash$)

Theorem 7. *If S is consistent and Harrop, then $Cn(S)$ is saturated.*

Proof. Let S be consistent and Harrop. By Theorem 5 (4), $Cn(S)$ is consistent. For join-primeness, let $a \vee b \in Cn(S)$. By Theorem 4, either $a \in Cn(S)$ or $b \in Cn(S)$, as required. Closure under $\vdash$ follows from Lemma 1 (11). □

The relationship between maximal consistent set (in the classical sense) and saturated set (in the intuitionistic sense) may be described as follows. A set of wff's is said to be maximal consistent if it is consistent and none of its proper extensions is consistent. In classical logic, this definition can be rephrased (equivalently) using the notion of "$\neg$-completeness". A set S of wff's is said to be $\neg$-complete, whenever $a \in S$ or $\neg a \in S$ for all wff a. Call S maximal consistent* if it is consistent and $\neg$-complete.[3]

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[3] We use the superscript * to emphasize that this version of maximal consistency is, to some extent, peculiar to classical logic.

Example 1. Suppose the language has two propositional letters a and b only. By Theorem 7, $Cn(a)$ is a saturated set. However, $Cn(a)$ is not $\neg$-complete, since neither $b \in Cn(a)$ nor $\neg b \in Cn(a)$. Therefore, not all saturated sets are maximal consistent*.

Theorem 8 shows that, for classical logic, the notion of saturated set coincides with that of a maximal consistent one.

Theorem 8. *The following applies:*

1. *Any maximal consistent* set S is saturated;*
2. *Suppose the law of excluded middle holds, i.e. $\vdash a \vee \neg a$. Any saturated set S is $\neg$-complete, and thus maximal consistent*.*

Proof. For the first claim, let S be maximal consistent*. For join-primeness, assume $a \vee b \in S$, but $a \notin S$ and $b \notin S$. By $\neg$-completeness, $\neg a \in S$ and $\neg b \in S$. The reader may easily verify that the set $\{a \vee b, \neg a, \neg b\}$ is not satisfiable in any model, and so (by Theorem 6) it is inconsistent. So, by Theorem 5 (3), S is inconsistent, contrary to the opening assumption. For closure under $\vdash$, assume $S \vdash a$ but $a \notin S$. By $\neg$-completeness, $\neg a \in S$. By (**Ref**), $S \vdash \neg a$. This is enough to make S inconsistent, given ($\neg$:**E**).

The second claim is [13, Lemma 5.6]. By (**Mon**), $S \vdash a \vee \neg a$. By Definition 2 (3), $a \vee \neg a \in S$. By Definition 2 (2), either $a \in S$ or $\neg a \in S$. □

The following will come in handy.

Theorem 9. *For any saturated set S,*

$$S = Cn(S) \tag{1}$$

$$\textit{If } a, b \in S, \textit{ then } a \wedge b \in S \tag{2}$$

Proof. For (1). The right-in-left inclusion is Definition 2 (3). The left-in-right inclusion is (**Ref**).

For (2), let $a, b \in S$. By (**Ref**), $S \vdash a$ and $S \vdash b$. By ($\wedge$:**I**), $S \vdash a \wedge b$. By Definition 2 (3), $a \wedge b \in S$. □

An analog of so-called Lindenbaum's lemma is available.

Theorem 10. *If $\Gamma \nvdash a$, then there is a saturated set S such that $\Gamma \subseteq S$ and $a \notin S$*

Proof. This is [14, Lemma 1]. □

out_2 can be reformulated as follows. To avoid proliferation of subscripts, we use the exact same name as in the original framework.

Definition 3 (out_2, intuitionistic basic output).

$$out_2(G,A) = \begin{cases} \cap\{Cn(G(S)) : A \subseteq S, S \text{ saturated}\}, & \text{if Con}A \\ Cn(h(G)), & \text{otherwise} \end{cases}$$

where $h(G)$ is the set of all heads of elements of G, viz. $h(G) = \{x : (a,x) \in G \text{ for some } a\}$.

The definition is well-behaved, because of Theorem 10. It guarantees that there is at least one saturated set extending A, when A is consistent.

A similar remark applies to out_4, which may be redefined thus:

Definition 4 (out_4, intuitionistic basic reusable output).

$$out_4(G,A) = \begin{cases} \cap\{Cn(G(S)) : A \subseteq S \supseteq G(S), S \text{ saturated}\}, & \text{if Con}A \\ Cn(h(G)), & \text{otherwise} \end{cases}$$

3 Completeness Results

This section gives completeness results for the first three intuitionistic output operations described in the previous section. To help with cross-reference, we shall refer to the completeness proof given in [9] as the original proof or the classical case. From now onwards it is understood that the output operations are defined on top of intuitionistic logic. We give the full details even when the argument is a re-run of the original one. This, in order to pinpoint what elementary rules are needed, and where.

The original completeness proofs for out_2 and out_3 both make essential use of the completeness result for out_1. We start by noticing that the latter one carries over to the intuitionistic setting. The proof for the classical case is outlined in [9]; we give it in full detail to make clear that it goes through in the intuitionistic case.

Theorem 11. *out_1 validates the rules of $deriv_1$.*

Proof. The verification is easy, and is omitted. Note that the argument for (AND) uses ($\wedge$:**I**). □

Theorem 12 (Soundness, simple-minded). $deriv_1(G,A) \subseteq out_1(G,A)$

Proof. Assume $x \in deriv_1(G,A)$. By definition, $x \in deriv_1(G,a)$ for some conjunction $a = a_1 \wedge ... \wedge a_n$ of elements of A. We need to show $x \in out_1(G,a)$. (By Lemma 1 (6), this is equivalent to $x \in out_1(G,\{a_1,...,a_n\})$, from which $x \in out_1(G,A)$ follows by monotony in A.) The proof is by induction on the length n of the derivation.

We give the base case $n = 1$ in full detail in order to highlight what elementary rules are needed. In this case, either $(a,x) \in G$ or (a,x) is the pair $(\top,\top)$. Suppose $(a,x) \in G$. By (**Ref**) $a \vdash a$, and thus $x \in G(Cn(a))$. By (**Ref**) again $G(Cn(a)) \vdash x$, which suffices for $x \in out_1(G,a)$. Suppose (a,x) is the pair $(\top,\top)$. By ($\top$:**I**), $G(Cn(\top)) \vdash \top$, which suffices for $\top \in out_1(G,\top)$.

The inductive case is straightforward, using Theorem 11. Details are omitted. □

Theorem 13 (Completeness, simple-minded). $out_1(G,A) \subseteq deriv_1(G,A)$

Proof. Let $x \in Cn(G(Cn(A)))$. We break the argument into cases depending on whether some elements of G are "triggered" or not.

Suppose $G(Cn(A)) = \emptyset$. By Lemma 1 (8), $\top \vdash x$. And by ($\top$:**I**) $a \vdash \top$ for some (arbitrarily chosen) conjunction $a = a_1 \wedge ... \wedge a_n$ of elements of A. The required derivation of (A,x) is shown below.

$$\dfrac{\dfrac{(\top,\top)}{(\top,x)}\,\text{WO}}{(a,x)}\,\text{SI}$$

Suppose $G(Cn(A) \neq \emptyset$. By (**C**) and Lemma 1 (6), there are $x_1,...,x_n (n > 0)$ in $G(Cn(A))$ such that $x_1 \wedge ... \wedge x_n \vdash x$. So G contains the pairs (a_1,x_1), ..., (a_n,x_n) with $A \vdash a_1$, ..., and $A \vdash a_n$. By ($\wedge$:**I**), $A \vdash a_1 \wedge ... \wedge a_n$. By (**C**) and Lemma 1 (6), $a \vdash a_1 \wedge ... \wedge a_n$ for some conjunction $a = b_1 \wedge ... \wedge b_m$ of elements of A. By definition, each (a_i,x_i) is derivable from G. Furthermore, by ($\wedge$:**E**) $a \vdash a_i$ for all $1 \leq i \leq n$. Based on this one might get a derivation of (A,x) from G as shown below.

$$\text{WO}\,\dfrac{\dfrac{\dfrac{(a_1,x_1)}{(a,x_1)}\,\text{SI} \qquad \ldots\ldots \qquad \dfrac{(a_n,x_n)}{(a,x_n)}\,\text{SI}}{(a,x_1 \wedge \ldots \wedge x_n)}\,\text{AND}}{(a,x)}$$

This completes the proof. $\square$

The argument for out_2 is more involved, and needs to be adapted.

Theorem 14. *out_2 validates all the rules of $deriv_2$.*

Proof. For (WO) and (AND), the verification is straightforward, and is omitted.

For (SI), assume $x \in out_2(G,a)$ and $b \vdash a$. Assume Incon$\{a\}$. By Theorem 5 (2), Incon$\{b\}$, and thus we are done. Assume Con$\{a\}$ but Incon$\{b\}$. In this case, we are done too, because

$$x \in \cap\{Cn(G(S)) : \{a\} \subseteq S, S \text{ saturated}\} \subseteq Cn(G(h(G)))$$

Assume Con$\{a\}$ and Con$\{b\}$. Consider a saturated set S such that $b \in S$. By (**Ref**), $S \vdash b$. By (**Mon**), $S, b \vdash a$, and thus, by (**Cut**), $S \vdash a$. By Definition 2 (3), $a \in S$. Thus, $x \in Cn(G(S))$, which suffices for $x \in out_2(G,b)$.

For (OR), assume $x \in out_2(G,a)$ and $x \in out_2(G,b)$. Assume Incon$\{a \vee b\}$. In this case, by Theorem 5 (1), Incon$\{a\}$ and Incon$\{b\}$, and so we are done. Suppose Con$\{a \vee b\}$. In this case, by Theorem 5 (1) again, either Con$\{a\}$ or Con$\{b\}$. Suppose the first, but not the second, applies (the argument is similar the other way around, and it still goes through with a minor adjustment if both apply). Consider a saturated set S such that $a \vee b \in S$. By Definition 2 (2) either $a \in S$ or $b \in S$. The latter is ruled out by the consistency of S and Theorem 5 (3). So $a \in S$. From Con$\{a\}$ and $x \in out_2(G,a)$, it follows that $x \in Cn(G(S))$. Thus, $x \in out_2(G, a \vee b)$, as required. $\square$

Corollary 1 (Soundness, basic output). $deriv_2(G,A) \subseteq out_2(G,A)$

Proof. By induction on the length of the derivation using Theorem 14. □

Theorem 15 (Completeness, basic output). $out_2(G,A) \subseteq deriv_2(G,A)$

Proof. Like in the original proof, we break the argument into cases. The first is a borderline case, and the second is the principal case.

For ease of exposition, we write (SI,AND) to indicate an application of SI followed by that of AND, and similarly for other rules. Thus, (SI,AND) abbreviates the following derived rule

$$\text{(SI,AND)}\ \frac{(a_1,x_1) \qquad \dots \qquad (a_n,x_n)}{(\wedge_{i=1}^n a_i, \wedge_{i=1}^n x_i)}$$

Case 1: Incon$A$. In this case, $out_2(G,A) = Cn(h(G))$. Let $x \in Cn(h(G))$. By (**C**), there are finitely many x_i's ($i \leq n$) in $h(G)$ such that $x \in Cn(x_1,\dots,x_n)$. A derivation of (a,x) from G may, then, be obtained as shown below, where a is the conjunction of a finite number of elements in A, and all the pairs (b_i,x_i) are in G.

$$\dfrac{\dfrac{\dfrac{(b_1,x_1)\quad \dots \quad \dots \quad (b_n,x_n)}{(\wedge_{i=1}^n b_i, \wedge_{i=1}^n x_i)}\,\text{(SI,AND)} \qquad \dfrac{x_1,\dots,x_n \vdash x}{\wedge_{i=1}^n x_i \vdash x}\,\text{Lem 1 (6)}}{(\wedge_{i=1}^n b_i, x)}\,\text{WO} \qquad \dfrac{\dfrac{\text{Incon}A}{A \vdash \wedge_{i=1}^n b_i}}{a \vdash \wedge_{i=1}^n b_i}\,\mathbf{C} + \text{Lem 1 (6)}}{(a,x)}\,\text{SI}$$

Case 2: Con$A$. Assume (for reductio) that $x \in out_2(G,A)$ but $x \notin deriv_2(G,A)$. From the former, we get that $x \in Cn(h(G))$. We use the same kind of maximality argument as in the original proof to derive the contradiction that $x \notin out_2(G,A)$.

We start by showing that A can be extended to some "maximal" $S \supseteq A$ such that $x \notin deriv_2(G,S)$. By maximal, we mean that $\forall S' \supset S$, $x \in deriv_2(G,S')$. Thus, S is amongst the "biggest" input sets S containing A and not making x derivable.

S is built from a sequence of sets $S_0, S_1, S_2, \dots$ as follows. Consider an enumeration $x_1, x_2, x_3, \dots$ of all the formulae. Put $S_0 = A$, and

$$S_n = \begin{cases} S_{n-1} \cup \{x_n\}, & \text{if } x \notin deriv_2(G, S_{n-1} \cup \{x_n\}) \\ S_{n-1}, & \text{otherwise} \end{cases}$$

It is straightforward to show by induction that

Claim 1. $x \notin deriv_2(G,S_n)$, for all $n \geq 0$.

Now we define S to be the infinite collection of all the wff's in any of the sets in the sequence:

$$S = \cup\{S_n : n \geq 0\}$$

Note that S includes each of the sets in the sequence:

Claim 2. $S_n \subseteq S$, for all $n \geq 0$.

So in particular S includes A (=S_0).

Claim 3. $A \subseteq S$.

Thus, S is an extension of A. To show that $x \notin deriv_2(G,S)$, three more results are needed; their proof is omitted.

Claim 4. $S_k \subseteq S_n$, for $k \leq n \geq 0$.

Claim 5. $x_k \in S_k$, whenever $x_k \in S$, for $k > 0$.

Claim 6. For every finite subset S' of S, $S' \subseteq S_n$, for some $n \geq 0$.

With these results in hand, the argument for $x \notin deriv_2(G,S)$ may run as follows. Assume, to reach a contradiction, that $x \in deriv_2(G,S)$. By compactness for $deriv_2$, $x \in deriv_2(G,S')$ for some finite $S' \subseteq S$. By Claim 6, $S' \subseteq S_n$ for some $n \geq 0$. By monotony in the right argument, $x \in deriv_2(G,S_n)$. This contradicts Claim 1.

Next, we show that S is maximal. Let $y \notin S$. Any such y is such that $y = x_n$, for some $n \geq 1$. By Claim 2 $S_n \subseteq S$, and thus $y \notin S_n$. By construction, $S_{n-1} = S_n$, and $x \in deriv_2(G,S_{n-1} \cup \{y\}) = deriv_2(G,S_n \cup \{y\})$. By Claim 2, $S_n \cup \{y\} \subseteq S \cup \{y\}$. By monotony in the right argument for $deriv_2$, we get that $x \in deriv_2(G,S \cup \{y\})$, as required.

Now, we show that S is a saturated set. This amounts to showing that S is i) consistent ii) closed under consequence, and iii) join-prime.

For i), assume $\text{Incon}\,S$. We have $x \in Cn(h(G))$. Note that $x \notin Cn(\emptyset)$. Otherwise, for the reason explained in the proof of Theorem 13, (A,x) would be derivable from G:

$$\dfrac{\dfrac{(\top,\top)}{(\top,x)}\ \text{WO}}{(a,x)}\ \text{SI}$$

Here a denotes the conjunction of finitely many elements of A (their choice may be arbitrary).

By (**C**) and Lemma 1 (6), it follows that there are $x_1,...,x_n (n > 0)$ in $h(G)$ such that $\wedge_{i=1}^{n} x_i \vdash x$. Let $a_1,...,a_n$ be the bodies of the rules in question. Since $\text{Incon}\,S$, $S \vdash \wedge_{i=1}^{n} a_i$. By (**C**) and Lemma 1 (6), $\wedge_{i=1}^{m} s_i \vdash \wedge_{i=1}^{n} a_i$ for $s_1,...,s_m \in S$. The following indicates how a derivation of (S,x) may be obtained from G, contradicting the result $x \notin deriv_2(G,S)$ previously established.

$$\text{(SI,AND)}\ \dfrac{\dfrac{\dfrac{(a_1,x_1)....(a_n,x_n)}{(\wedge_{i=1}^{n} a_i, \wedge_{i=1}^{n} x_i)}}{(\wedge_{i=1}^{n} a_i, x)}\ \text{WO}}{(\wedge_{i=1}^{m} s_i, x)}\ \text{SI}$$

For ii), let $S \vdash y$ and $y \notin S$. From the former, $\wedge_{i=1}^{n} s_i \vdash y$, for $s_1,...,s_n \in S$ by (**C**) and Lemma 1 (6). From the latter, $x \in deriv_2(G,S \cup \{y\})$, by maximality of S. This means that the pair $(\wedge_{i=1}^{m} a_i \wedge y, x)$ can be derived from G, with $a_1,...,a_m \in S$. From this, the contradiction $x \in deriv_2(G,S)$ follows:

$$\dfrac{\begin{array}{c}\vdots\\(\wedge_{i=1}^{m} a_i\wedge y,x)\end{array}\qquad \dfrac{\dfrac{\dfrac{\wedge_{i=1}^{n} s_i\vdash y}{\wedge_{i=1}^{n} s_i,\wedge_{i=1}^{m} a_i\vdash y}\,\textbf{(Mon)}}{\wedge_{i=1}^{n} s_i\wedge(\wedge_{i=1}^{m} a_i)\vdash y}\,\text{Lem 1(6)}\qquad \dfrac{\dfrac{\wedge_{i=1}^{m} a_i\vdash\wedge_{i=1}^{m} a_i}{\wedge_{i=1}^{n} s_i,\wedge_{i=1}^{m} a_i\vdash\wedge_{i=1}^{m} a_i}\,\textbf{(Mon)}}{\wedge_{i=1}^{n} s_i\wedge(\wedge_{i=1}^{m} a_i)\vdash\wedge_{i=1}^{m} a_i}\,\text{Lem 1(6)}}{\wedge_{i=1}^{n} s_i\wedge(\wedge_{i=1}^{m} a_i)\vdash\wedge_{i=1}^{m} a_i\wedge y}\,(\wedge{:}\mathbf{I})}{(\wedge_{i=1}^{n} s_i\wedge(\wedge_{i=1}^{m} a_i),x)}\,\text{SI}$$

For iii), let $a\vee b\in S$. Assume $a\notin S$ and $b\notin S$. Any such a and b are such that $a=x_n$, for some $n\geq 1$, and $b=x_m$, for some $m\geq 1$. By construction, $x\in deriv_2(G,S_n\cup\{a\})$ and $x\in deriv_2(G,S_m\cup\{b\})$. By Claim 4 either $S_n\subseteq S_m$ or $S_m\subseteq S_n$. Suppose the first applies (the argument for the other case is similar). By monotony in the right argument for $deriv_2$, $x\in deriv_2(G,S_m\cup\{a\})$. This means that the pair $(\wedge_{i=1}^{l} s_i\wedge a,x)$ can be derived from G, with $s_1,...,s_l\in S_m$, and that the pair $(\wedge_{i=1}^{p} a_i\wedge b,x)$ can be derived from G, with $a_1,...,a_p\in S_m$. Note that, by Lemma 1 (10), we have

$$\wedge_{i=1}^{l} s_i\wedge(\wedge_{i=1}^{p} a_i)\wedge(a\vee b)\vdash(\wedge_{i=1}^{l} s_i\wedge a)\vee(\wedge_{i=1}^{p} a_i\wedge b)$$

Thus,

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\(\wedge_{i=1}^{l} s_i\wedge a,x)\end{array}\qquad\begin{array}{c}\vdots\\(\wedge_{i=1}^{p} a_i\wedge b,x)\end{array}}{((\wedge_{i=1}^{l} s_i\wedge a)\vee(\wedge_{i=1}^{p} a_i\wedge b),x)}\,\text{OR}}{(\wedge_{i=1}^{l} s_i\wedge(\wedge_{i=1}^{p} a_i)\wedge(a\vee b),x)}\,\text{SI}$$

So, $x\in deriv_2(G,S_m\cup\{a\vee b\})$. Since $a\vee b\in S$ and $S_m\subseteq S$, $S_m\cup\{a\vee b\}\subseteq S$. So by monotony $x\in deriv_2(G,S)$, a contradiction.

This concludes the main step of the proof. The final step is as in the original proof. We have $out_1(G,S)=deriv_1(G,S)\subseteq deriv_2(G,S)$. From $x\notin deriv_2(G,S)$, it then follows that $x\notin out_1(G,S)=Cn(G(Cn(S)))$. Since S is saturated, $S=Cn(S)$ by Theorem 9 (1). So $x\notin Cn(G(S))$. Furthermore, A is consistent, and $S\supseteq A$. By Definition 3, $x\notin out_2(G,A)$, and the proof may be considered complete. □

It is noteworthy that the argument for saturatedness uses Lemma 1 (10), which in turn appeals to the deduction theorem.

It is also interesting to see what happens if, in Definition 3, maximal consistency is used in place of saturatedness. We would need to establish that S is maximal consistent. However, the latter fact does not follow from the stated hypotheses, and what has already been established. Therefore, the proof does not go through. If maximal consistency* is used instead, then the proof does not go through either unless the base logic has the law of excluded middle. The latter law is needed to establish that S is $\neg$-complete. To see why, suppose there is some a such that $a\notin S$ and $\neg a\notin S$. A similar argument as in the proof for saturatedness yields $x\in deriv_2(G,S_m\cup\{a\})$ and $x\in deriv_2(G,S_m\cup\{\neg a\})$, from which $x\in deriv_2(G,S_m\cup\{a\vee\neg a\})$ follows.

However, in the absence of excluded middle, there is no guarantee that $a \vee \neg a \in S$, and so the required contradiction $x \in deriv_2(G,S)$ does not follow any more.

We now turn to the reusable operation out_3. In [9], the soundness and completeness results are established for a singleton input set A. We extend the argument to an input set A of arbitrary cardinality.

Theorem 16 (Soundness and completeness, simple-minded reusable output). $out_3(G,A) = deriv_3(G,A)$.

Proof. For the soundness part, we only verify that the new rule (CT) is still valid. Let $x \in out_3(G,a)$ and $y \notin out_3(G,a)$. To show: $y \notin out_3(G,a \wedge x)$. From the second hypothesis, there is some $B = Cn(B) \supseteq G(B)$ with $a \in B$ and $y \notin Cn(G(B))$. From the first hypothesis, $x \in Cn(G(B))$. By (**Mon**), $Cn(G(B)) \subseteq Cn(B)$. So $x \in Cn(B)$. By (**Ref**), $a \in Cn(B)$. By ($\wedge$:**I**), $a \wedge x \in Cn(B)$. So, $a \wedge x \in B$. This means that $y \notin out_3(G,a \wedge x)$ as required.

For the completeness part, we argue contrapositively. Let $x \notin deriv_3(G,A)$. To show: $x \notin out_3(G,A)$.

Let $B = Cn(A \cup deriv_3(G,A))$. By (**Ref**), $A \subseteq B$. By Lemma 1 (11), $Cn(B) = B$. To show: i) $G(B) \subseteq B$; ii) $x \notin Cn(G(B))$.

For i). Let $y \in G(B)$. So $(b,y) \in G$ for some $b \in B$. Hence $A \cup deriv_3(G,A) \vdash b$. By (**C**), $a_1,...,a_m,x_1,...,x_n \vdash b$ for $a_1,...,a_m \in A$ and $x_1,...,x_n \in deriv_3(G,A)$. For all $i \leq n$, $x_i \in deriv_3(G,b_i)$, where b_i is the conjunction of elements in A. By Lemma 1 (6), $\wedge_{i=1}^m a_i, \wedge_{i=1}^n x_i \vdash b$. By ($\rightarrow$:**I**), $\wedge_{i=1}^n x_i \vdash \wedge_{i=1}^m a_i \rightarrow b$. Based on this, one might argue that $y \in deriv_3(G,A)$ as follows. By Lemmas 1 (2) and (9), $\wedge_{i=1}^n b_i \wedge (\wedge_{i=1}^m a_i) \wedge (\wedge_{i=1}^m a_i \rightarrow b) \vdash b$ and $\wedge_{i=1}^n b_i \wedge (\wedge_{i=1}^m a_i) \vdash \wedge_{i=1}^n b_i$. We, thus, have:

$$\frac{\dfrac{\dfrac{\dfrac{\vdots}{(b_1,x_1)} \quad \dots \quad \dots \quad \dfrac{\vdots}{(b_n,x_n)}}{(\wedge_{i=1}^n b_i, \wedge_{i=1}^n x_i)}\,\text{(SI,AND)}}{(\wedge_{i=1}^n b_i \wedge (\wedge_{i=1}^m a_i), \wedge_{i=1}^m a_i \rightarrow b)}\,\text{(SI,WO)} \qquad \dfrac{(b,y)}{(\wedge_{i=1}^n b_i \wedge (\wedge_{i=1}^m a_i) \wedge (\wedge_{i=1}^m a_i \rightarrow b), y)}\,\text{SI}}{(\wedge_{i=1}^n b_i \wedge (\wedge_{i=1}^m a_i), y)}\,\text{CT}$$

Once this established, the conclusion $y \in B$ immediately follows from (**Ref**).

The argument for ii) uses a *reductio ad absurdum*, and invokes the completeness result for out_1. Assume $x \in Cn(G(B))$. This amounts to assuming that $x \in out_1(G,A \cup deriv_3(G,A))$. By the above completeness result, $x \in deriv_1(G,A \cup deriv_3(G,A))$. By definition of $deriv_1$, $x \in deriv_1(G, \wedge_{i=1}^n a_i \wedge (\wedge_{i=1}^m x_i))$, where $a_1,...,a_n \in A$ and x_1, ..., $x_m \in deriv_3(G,A)$. So, *a fortiori*, $x \in deriv_3(G, \wedge_{i=1}^n a_i \wedge (\wedge_{i=1}^m x_i))$. For all $i \leq m$, $x_i \in deriv_3(G,b_i)$, where b_i is the conjunction of elements in A. The following indicates how a derivation of (A,x) may be obtained, contradicting the opening assumption.

$$\dfrac{\dfrac{\overset{\vdots}{(b_1,x_1)} \quad \dots \quad \dots \quad \overset{\vdots}{(b_m,x_m)}}{(\wedge_{i=1}^m b_i \wedge (\wedge_{i=1}^n a_i), \wedge_{i=1}^m x_i)}\,(\mathrm{SI,AND}) \qquad \dfrac{\dfrac{\vdots}{(\wedge_{i=1}^n a_i \wedge (\wedge_{i=1}^m x_i), x)}}{(\wedge_{i=1}^m b_i \wedge (\wedge_{i=1}^n a_i)\wedge(\wedge_{i=1}^m x_i), x)}\,\mathrm{SI}}{(\wedge_{i=1}^m b_i \wedge (\wedge_{i=1}^n a_i), x)}\,\mathrm{CT}$$

This shows that $x \notin out_3(G,A)$. □

It is noteworthy that the argument for i) uses both (→:**I**) and (→:**E**).

We end with a few remarks on out_4. The reader may easily verify that the four rules of $deriv_4$ are still validated. However, the original completeness proof breaks down when going intuitionistic. This is because what is described as Lemma 11 a in the original argument is no longer provable. This is the lemma that states that $deriv_4(G, a \wedge (b \to x)) \subseteq deriv_4(G,a)$ whenever $(b,x) \in G$. Its proof uses the following law, which is no longer available:

(†) $$a \vdash (a \wedge b) \vee (a \wedge (b \to x))$$

This brings to the fore what may be considered the bottom-line between the two approaches. In Makinson and van der Torre [9] there is a brief mention of the rule of (as they call it) "ghost contraposition". This is the rule

$$(\mathrm{GC})\ \dfrac{(a,b) \qquad (\neg a, c)}{(\neg c, b)}$$

Intuitively: although we cannot contrapose the rightmost premisse $(\neg a, c)$, we can still use its contrapositive $(\neg c, a)$ for an application of the rule of transitivity. If $(\neg a, c)$ is rewritten as $(\neg c, a)$, then the conclusion $(\neg c, b)$ follows by plain transitivity.

It is not difficult to see that although none of the output operations validate plain contraposition (GC) holds for the classically-based reusable basic output out_4. In this respect, contraposition still plays a 'ghostly' role for out_4. The following shows how to derive (GC).

$$\dfrac{\dfrac{\dfrac{\dfrac{(\neg a, c)}{(\neg c \wedge \neg a, c)}\,\mathrm{SI} \qquad \dfrac{(a,b)}{(\neg c \wedge \neg a \wedge c, b)}\,\mathrm{SI}}{(\neg c \wedge \neg a, b)}\,\mathrm{CT} \qquad \dfrac{(a,b)}{(\neg c \wedge a, b)}\,\mathrm{SI}}{((\neg c \wedge \neg a) \vee (\neg c \wedge a), b)}\,\mathrm{OR}}{(\neg c, b)}\,\mathrm{SI}$$

In an intuitionistic setting the last move is blocked, because (‡) does not hold:

(‡) $$\neg c \vdash (\neg c \wedge \neg a) \vee (\neg c \wedge a)$$

If (‡) was valid, then we would get the law of excluded middle. This can be seen as follows. Substituting, in (‡), $x \wedge \neg x$ for c, we get by Theorem 2:

$$\neg(x \wedge \neg x) \vdash (\neg(x \wedge \neg x) \wedge \neg a) \vee (\neg(x \wedge \neg x) \wedge a)$$

The reader may easily verify that $\vdash \neg(x \wedge \neg x)$. So,

$$\vdash (\neg(x \wedge \neg x) \wedge \neg a) \vee (\neg(x \wedge \neg x) \wedge a)$$

This in turn can be simplified into:[4]

$$\vdash a \vee \neg a$$

Example 2 provides a counter-model to (‡), which is *mutatis mutandis* a counter-model to the law of excluded middle.

Example 2. Consider a model $M = (W, \geq, V)$ with $W = \{s,t\}$, $s \geq s$, $t \geq t$, $t \geq s$, $V(a) = \{t\}$, and $V(c) = \emptyset$. This can be depicted as in Figure 1. Here the general convention is that $v \geq u$ iff either $u = v$ or v is above u. And each world is labelled with the atoms it makes true. Thus, a missing atom indicates falsehood. s forces $\neg c$,

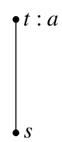

Fig. 1 A counter-model to (‡)

because neither s nor t forces c. s does not force $\neg c \wedge \neg a$, because it does not force $\neg a$ (witness: t). And neither does s force $\neg c \wedge a$. Therefore, $(\neg c \wedge \neg a) \vee (\neg c \wedge a)$ is not a semantic consequence of $\neg c$, and thus (by soundness) the former is not derivable from the latter.

As the reader may see, the fact that $\neg$ occurs in the premiss in (‡) plays no role. This is also a counter-model to the law of excluded middle. Indeed, $a \vee \neg a$ is not forced at s.

Example 3 shows that (GC) fails in an intuitionistic setting.

Example 3 (Ghost contraposition). Put $G = \{(a,b), (\neg a, c)\}$, where a, b and c are atomic formulae. Suppose out_4 is based on intuitionistic logic.

- Assume the input is $A = \{a\}$. Since a is consistent, the top clause in Definition 4 applies. Consider any saturated set S meeting the requirements mentioned in this clause. Any such S contains a. And, for any such S, $b \in G(S) \subseteq S$, so that $G(S) \vdash b$, by (**Ref**). This implies that $b \in out_4(G,a)$.
- Assume the input is $A = \{\neg a\}$. For a similar reason, $c \in out_4(G, \neg a)$.

[4] Using Theorem 3, $\top \dashv\vdash \neg(x \wedge \neg x)$, Lemma 1 (7) and $a \vee b \dashv\vdash b \vee a$.

- Assume the input is $A = \{\neg c\}$. Since $\neg c$ is consistent, the top clause in Definition 4 applies again. Consider $Cn(\neg c)$. By Theorem 7, $Cn(\neg c)$ is saturated. By (**Ref**), $\neg c \in Cn(\neg c)$. And $Cn(\neg c) \supseteq G(Cn(\neg c)) = \emptyset$. Furthermore, $b \notin Cn(\emptyset)$. So there is a saturated set S such that $A \subseteq S \supseteq G(S)$ but $b \notin Cn(G(S))$, which suffices for $b \notin out_4(G, \neg c)$.

The counter-example is blocked, if out_4 is defined in terms of maximal consistent* rather than saturated sets. Given input $\neg c$, there is only one maximal consistent* set S meeting the requirement $A \subseteq S \supseteq G(S)$. It is $Cn(a, \neg c, b)$. So $b \in out_4(G, \neg c)$.

Of course, this leaves open the question of whether a representation result may, or may not, be obtained for intuitionistic out_4.

At first sight, it would seem that out_4 is more suitable to model normative reasoning, because of the presence in its proof-theory of rules that are quite attractive. Indeed the operation supports both reasoning by cases (OR) and chaining (CT). But one would like to have the core rules of $deriv_4$ without getting (GC) by the same way. Indeed, the following natural language example, due to Hansen [4], suggests that failure of ghost contraposition is a desirable feature. Let a, b, and c stand for the propositions that it is raining, that I wear my rain coat, and that I wear my best suit, respectively. It makes sense for my mum to order me to wear my rain coat if it rains, and my best suit if it does not. It does not follow that I am obliged to wear my rain coat given that I cannot wear my best suit (e.g., it is in the laundry).

4 Conclusion and Future Research

This paper has demonstrated that, for three unconstrained output operations, the use of intuitionistic logic as base logic does not affect the axiomatic characterization of the resulting framework. This shows that it would be a mistake to think that I/O logic, in one way or another, 'presupposes' classical logic.

The proofs given in the paper help to appreciate what elementary rules are required besides the Tarski conditions. It is natural to ask if the full power of intuitionistic logic is needed for the completeness results to hold.

We said that intuitionistic logic is the smallest logic that allows for both the deduction theorem and the *ex falso* principle. The completeness proofs for out_2 and out_3 both invoke the former. But (as far as we can see) they do not appeal to the latter, and neither do the arguments for soundness. Therefore, it looks as if our results carry over to the minimal logic of Johansson [6], where the *ex falso* principle goes away. It is striking that the latter principle is used to show the hard half of the compactness theorem in terms of consistency. This is the right-to-left direction of the biconditional appearing in the statement of Theorem 5 (3). It states that a (possibly infinite) set of wffs is consistent if every finite subset of it is consistent. However, this property does not seem to play any role in the arguments.

For out_1, neither the deduction theorem nor the *ex falso* principle are used. Instead the argument makes essential use of ($\wedge$:**I**), ($\wedge$:**E**), and ($\top$:**I**). These rules can be viewed as the cornerstone of a natural deduction system, and they appear in logics weaker than intuitionistic logic, like the aforementioned minimal logic. What is clear is that some substructural logic like relevance logic would not do the job. ($\top$:**I**) goes away along with (**Mon**).

Several directions for future research can be taken. First, the question of whether a representation result may, or may not, be obtained for out_4 remains an open problem. Second, there are known embeddings of the classically-based out_2 and out_4 into modal logic. It would also be interesting to know if their intuitionistic analogs can be similarly mapped onto intuitionistic modal logic. Third, the question naturally arises as to whether one can move to the algebraic level by using lattices. Both classical and intuitionistic logics can be given an algebraic treatment. The interesting thing about lattices is that they give a geometrical flavor to I/O logic. A lattice is mainly about moving from points to points along a relation $\geq$. The travelling always goes in the upwards direction. We can think of the set G of generators used in I/O logic as "jump" points or bridges. A pair $(a,x) \in G$ is an instruction to deviate from a path that would be normally taken. Once we have reached node a, instead of continuing up we jump to an unrelated node x and continue your journey upwards.

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